Bayesian Unit-level Modeling of Categorical Survey Data with a Longitudinal Design

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Introduction

Overview

Motivation: Unit-level modeling for small area estimation has received significant attention in recent years, but important methodological gaps remain with

- Longitudinal designs
  - Household Pulse Survey (HPS)
  - Survey of Income and Program Participation (SIPP)
  - National Crime Victimization Survey, etc.
- Categorical responses,
  - Nominal (e.g., Type of health insurance, housing tenure)
  - Ordinal (e.g., Likert scale, frequency of symptoms)
- While accounting for survey design and high dimensionality

So far,

- Parker et al. (2022) cross-sectional binary and nominal models
- Vedensky et al. (2023) longitudinal Gaussian and binary models[1]

Gaps to fill in today:

- Longitudinal nominal
- Cross-sectional ordinal
- Longitudinal ordinal

[1] https://arxiv.org/abs/2304.07897v1

Background

Longitudinal design

Household Pulse was launched to track changes at the start of COVID pandemic and employed a rotating panel design

Panel / week	Panel 1	Panel 2	Panel 3	Panel 4
Week 1	Initial Full Sample			
Week 2	Respondents from week 1 only	New Sample		
Week 3	Respondents from week 2 only	Respondents from week 2 only	New Sample	
Week 4		Respondents from week 3 only	Respondents from week 3 only	...
Week 5			⋮	⋮

Table 1: HPS panel structure (adapted from HPS documentation)

Need to account for

- "temporal" correlation across weeks
- "longitudinal" correlation within person/household's responses

It is common for dependence to exist between response variable and probability of selection.

- Important to adjust for *informative sampling* (IS) in a unit-level model otherwise bias may occur.
- Parker et al. (2023) provide an overview of the problems that arise, as well as remedies.
- One remedy is to use a *pseudo-likelihood* (Binder, 1983; Skinner, 1989; Savitsky and Toth, 2016)

$$\text{PL}(\boldsymbol{y}|\boldsymbol{\theta}) = \prod_{i \in \mathcal{S}} f(y_i|\boldsymbol{\theta})^{w_i}.$$

Starting point: cross-sectional, binary unit-level model with Bayesian pseudo-likelihood (Parker et al., 2022)

$$\begin{aligned}
\boldsymbol{y}|\boldsymbol{\beta},\boldsymbol{\eta} &\propto \prod_{i\in S} \text{Binom}(y_i|n_i,p_i)^{\widetilde{w}_i} \\
\text{logit}(p_i) &= \boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{\psi}_i'\boldsymbol{\eta} \\
\boldsymbol{\eta}|\sigma_\eta^2 &\sim N_r(0_r, \sigma_\eta^2 \boldsymbol{I}_r) \\
\boldsymbol{\beta} &\sim N_q(0_q, \sigma_\beta^2 \boldsymbol{I}_q) \\
\sigma_\eta^2 &\sim \text{IG}(a,b) \\
\sigma_\beta, a, b &> 0.
\end{aligned}$$

- $\boldsymbol{\psi}_i$ may be an incidence vector or a spatial basis function.
- *Pólya-Gamma data augmentation* (Polson et al., 2013) leads to conjugacy

Nominal

Stick-breaking (Fienberg, 1980; Linderman et al., 2015)

For nominal responses with K unordered categories, can write

$$\text{Multinomial}((y_1, \ldots, y_K)|n, \boldsymbol{p}) = \prod_{k=1}^{K-1} \text{Binomial}(y_k|n_k, \tilde{p}_k),$$

where $n_k = n - \sum_{j<k} y_j$ and $\tilde{p}_k = p_k/(1 - \sum_{j<k} p_j)$.

That is, we can fit $K - 1$ binomial models then reconstruct $\boldsymbol{p}$ as

$$p_k = \tilde{p}_k \prod_{j<k} (1 - \tilde{p}_j).$$

Nominal unit-level model

(Parker et al., 2022)

A nominal model can then take the form

$$\boldsymbol{y}|\boldsymbol{\beta},\boldsymbol{\eta} \propto \prod_{i\in S}\prod_{k=1}^{K-1} \text{Binom}(y_{ik}|n_{ik},p_{ik})^{\widetilde{w}_i}$$

$$\text{logit}(\tilde{p}_{ik}) = \boldsymbol{x}_i'\beta_k + \boldsymbol{\psi}_i'\boldsymbol{\eta}_k$$

$$\boldsymbol{\eta}_k|\sigma^2_{\eta k} \sim N_r(0_r, \sigma^2_\eta \boldsymbol{I}_r), \quad k = 1,\ldots,K-1$$

$$\boldsymbol{\beta}_k \sim N_q(0_q, \sigma^2_\beta \boldsymbol{I}_q), \quad k = 1,\ldots,K-1$$

$$\sigma^2_{\eta k} \sim \text{IG}(a,b), \quad k = 1,\ldots,K-1$$

$$\sigma_\beta, a, b > 0.$$

Sequential ordinal models

(Tutz, 1990; Albert and Chib, 2001)

For ordered categories

- take our latent variable $\boldsymbol{\mu}_i = \boldsymbol{x}_i'\boldsymbol{\beta} + \boldsymbol{\psi}_i'\boldsymbol{\eta}$
- introduce unordered cutpoints $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{K-1})$
- suppose

$$P(y_i = k | y_i \geq k, \boldsymbol{\gamma}, \boldsymbol{\mu}_i) = F(\gamma_k - \boldsymbol{\mu}_i).$$

Then

$$P(y_i = k | \gamma_k, \boldsymbol{\mu}_i) = F(\gamma_k - \boldsymbol{\mu}_i) \prod_{j<k} (1 - F(\gamma_j - \boldsymbol{\mu}_i)).$$

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Then

$$\underbrace{P(y_i = k | \gamma_k, \boldsymbol{\mu}_i)}_{p_k} = \underbrace{F(\gamma_k - \boldsymbol{\mu}_i)}_{\tilde{p}_k} \prod_{j<k} (\underbrace{1 - F(\gamma_j - \boldsymbol{\mu}_i)}_{1-\tilde{p}_j}).$$

Ordinal unit-level model

Cross-sectional

Taking $F(\cdot) = \text{logit}^{-1}(\cdot)$ and placing a prior on $\boldsymbol{\gamma}$ leads to

$$\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{\eta}, \boldsymbol{\gamma} \propto \prod_{i \in \mathcal{S}} \prod_{k=1}^{K-1} \text{Binom}(y_{ik}|n_{ik}, \tilde{p}_{ik})^{\tilde{w}_i}$$

$$\text{logit}(\tilde{p}_{ik}) = \gamma_k - \boldsymbol{x}_i'\boldsymbol{\beta} - \boldsymbol{\psi}_i\boldsymbol{\eta}$$

$$\boldsymbol{\eta} \sim \text{N}_r(0, \sigma_\eta^2 I_r)$$

$$\boldsymbol{\beta} \sim \text{N}_q(0, \sigma_\beta^2 I_q)$$

$$\boldsymbol{\gamma} \sim \text{N}_{K-1}(0, \sigma_\gamma^2 I_{K-1})$$

$$\sigma_\eta^2 \sim \text{IG}(a, b)$$

- Pólya-Gamma data augmentation again yields conjugacy
- If $\tilde{w}_i = 1$ for all i, this is Bayesian ordinal logistic regression with a fully Gibbs sampler

Ordinal unit-level model

With time dependence

To capture time dependence, add AR(1) structure to random effects

$$\boldsymbol{y}|\boldsymbol{\beta},\boldsymbol{\eta},\boldsymbol{\gamma} \propto \prod_{t=1}^{T}\prod_{i\in\mathcal{S}_t}\prod_{k=1}^{K-1}\text{Binom}(y_{itk}|n_{itk},\tilde{p}_{itk})^{\tilde{w}_{it}}$$

$$\text{logit}(\tilde{p}_{itk}) = \boldsymbol{c}_{it}'\boldsymbol{\gamma}_k - \boldsymbol{x}_i'\boldsymbol{\beta} - \boldsymbol{\psi}_i\boldsymbol{\eta}_t$$

$$\boldsymbol{\gamma} \sim \text{N}_g(0,\sigma_\gamma^2 I_g)$$

$$\boldsymbol{\eta}_t|\boldsymbol{\eta}_{t-1},\phi,\sigma_\eta^2 \sim \text{N}_r(\phi\boldsymbol{\eta}_{t-1},\sigma_\eta^2 I_r),\quad t=2,\ldots,T$$

$$\boldsymbol{\eta}_1|\sigma_\eta^2 \sim \text{N}_r(0,\sigma_\eta^2 I_r)$$

$$\boldsymbol{\beta} \sim \text{N}_q(0,\sigma_\beta^2 I_q)$$

$$\phi \sim \text{Unif}(-1,1)$$

$$\sigma_\eta^2 \sim \text{IG}(a,b).$$

Ordinal unit-level model

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$$\boldsymbol{y}|\boldsymbol{\beta},\boldsymbol{\eta},\boldsymbol{\gamma} \propto \prod_{t=1}^{T}\prod_{i\in\mathcal{S}_t}\prod_{k=1}^{K-1}\text{Binom}(y_{itk}|n_{itk},\tilde{p}_{itk})^{\tilde{w}_{it}}$$

$$\text{logit}(\tilde{p}_{itk}) = \boldsymbol{c}_{it}'\boldsymbol{\gamma}_k - \boldsymbol{x}_i'\boldsymbol{\beta} - \boldsymbol{\psi}_i\boldsymbol{\eta}_t$$

$$\boldsymbol{\gamma} \sim \text{N}_g(0,\sigma_\gamma^2 I_g)$$

$$\boldsymbol{\eta}_t|\boldsymbol{\eta}_{t-1},\phi,\sigma_\eta^2 \sim \text{N}_r(\phi\boldsymbol{\eta}_{t-1},\sigma_\eta^2 I_r),\quad t=2,\ldots,T$$

$$\boldsymbol{\eta}_1|\sigma_\eta^2 \sim \text{N}_r(0,\sigma_\eta^2 I_r)$$

- Cutpoints vary with time
- Covariate with $K+1$ levels indexes whether prev. response = NA, $1,\ldots,K$

Nominal unit-level model

With time dependence

Similarly for the nominal model

$$\boldsymbol{y}|\boldsymbol{\beta},\boldsymbol{\eta} \propto \prod_{t=1}^{T}\prod_{i\in\mathcal{S}_t}\prod_{k=1}^{K-1} \text{Binomial}(y_{itk}|n_{itk},\tilde{p}_{itk})^{\widetilde{w}_{it}}$$

$$\text{logit}(\widetilde{p}_{itk}) = \boldsymbol{x}_i'\boldsymbol{\beta}_k + \boldsymbol{\psi}_i'\boldsymbol{\eta}_{tk}$$

$$\boldsymbol{\eta}_{tk}|\boldsymbol{\eta}_{(t-1)k},\phi_k,\sigma^2_{\eta k} \sim \text{N}_m(\phi_k\boldsymbol{\eta}_{(t-1)k},\sigma^2_{\eta k}I_m), \quad k=1,\ldots,K-1$$

$$\boldsymbol{\eta}_{1k}|\sigma^2_{\eta_1 k} \sim \text{N}_m(0,\sigma^2_{\eta_1 k}I_m), \quad k=1,\ldots,K-1$$

$$\boldsymbol{\beta}_k \sim \text{N}_q(0,\sigma^2_\beta I_q), \quad k=1,\ldots,K-1$$

$$\phi_k \sim \text{Unif}(-1,1)$$

$$\sigma^2_{\eta k}, \overset{ind.}{\sim} IG(a,b)$$

$$\sigma_\beta, a, b > 0.$$

Empirical simulation

Setup

To assess the proposed models, we conduct an empirical simulation study

- Treat entire HPS Phase 1 data as population
- Take 100 informative subsamples
- Fit models to each sample and generate predictions for population
- Compare direct estimates, cross-sectional, and longitudinal models

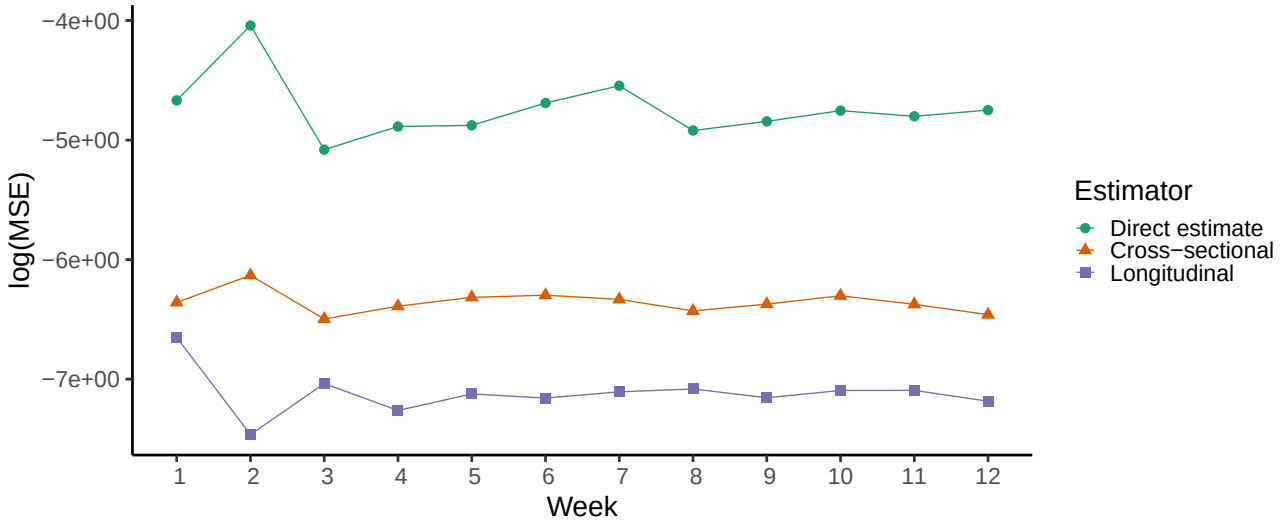
Empirical simulation

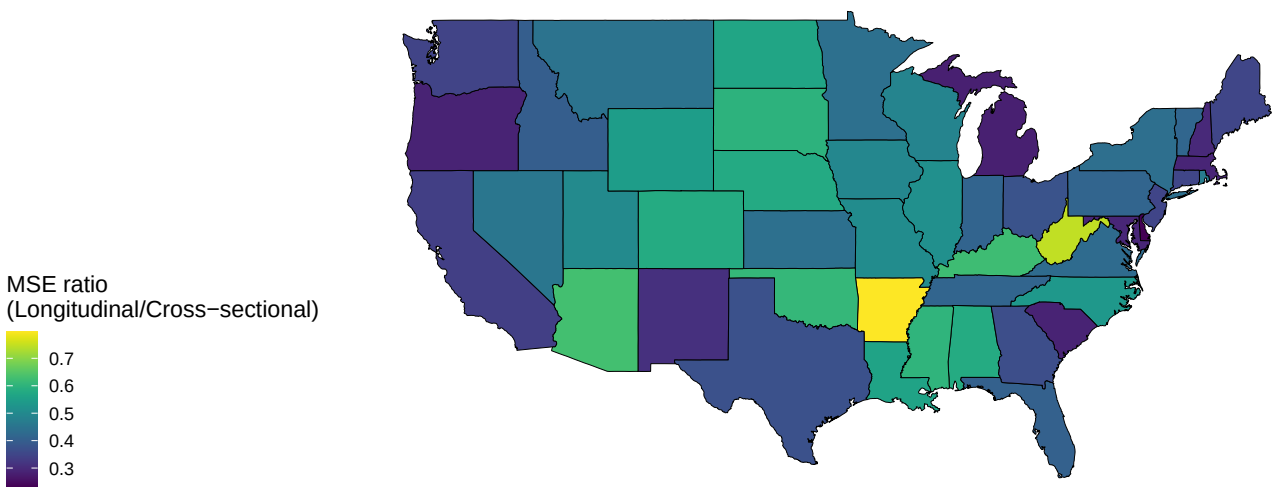
Setup

- Nominal response: "Is your house or apartment..."
  1. Owned free and clear?
  2. Owned with a mortgage or loan?
  3. Rented?
  4. Occupied without rent?
- Ordinal Response: Frequency of anxiety over previous 7 days...
  1. Not at all
  2. Several days
  3. More than half the days
  4. Nearly every day
- Covariates: sex, race, age, prev. response
- Take *probability proportional to size* sample with expected size 2% of the population ($N \approx 10^6$, both cases)

Empirical simulation

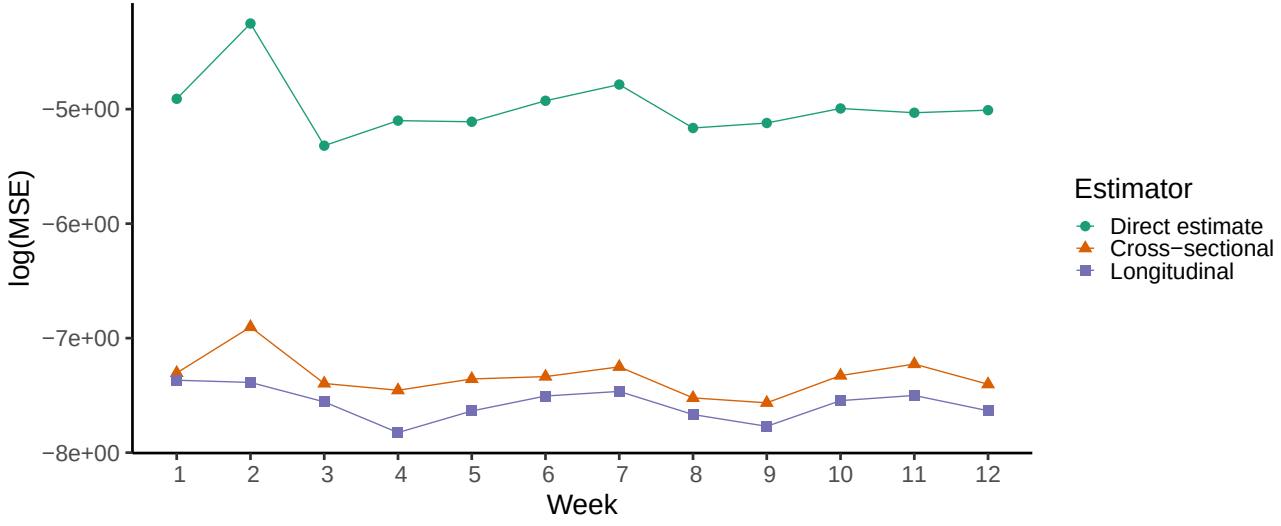
Nominal results

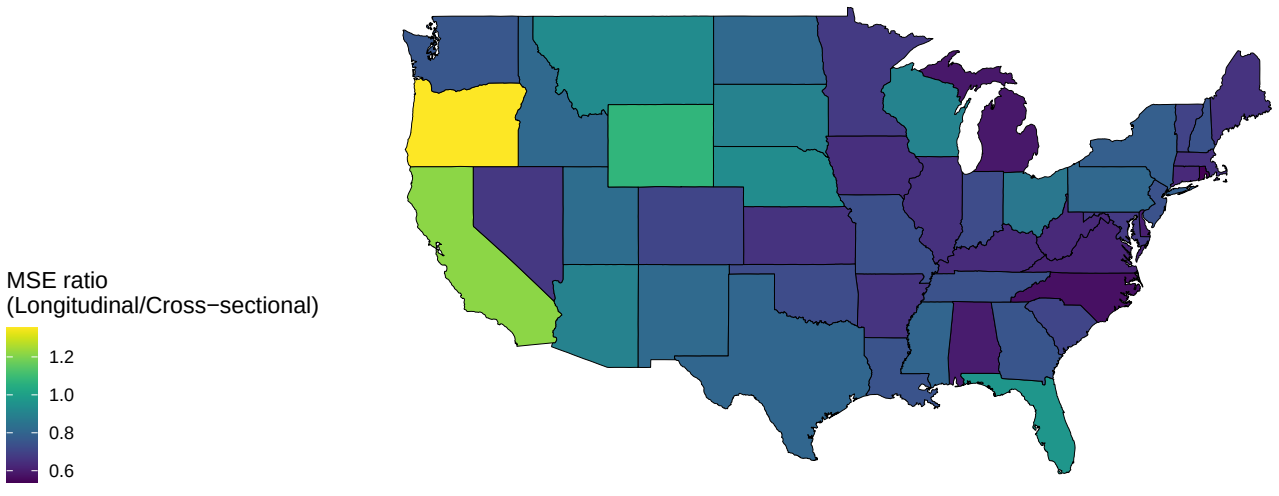




Empirical simulation

Ordinal results





Data analysis

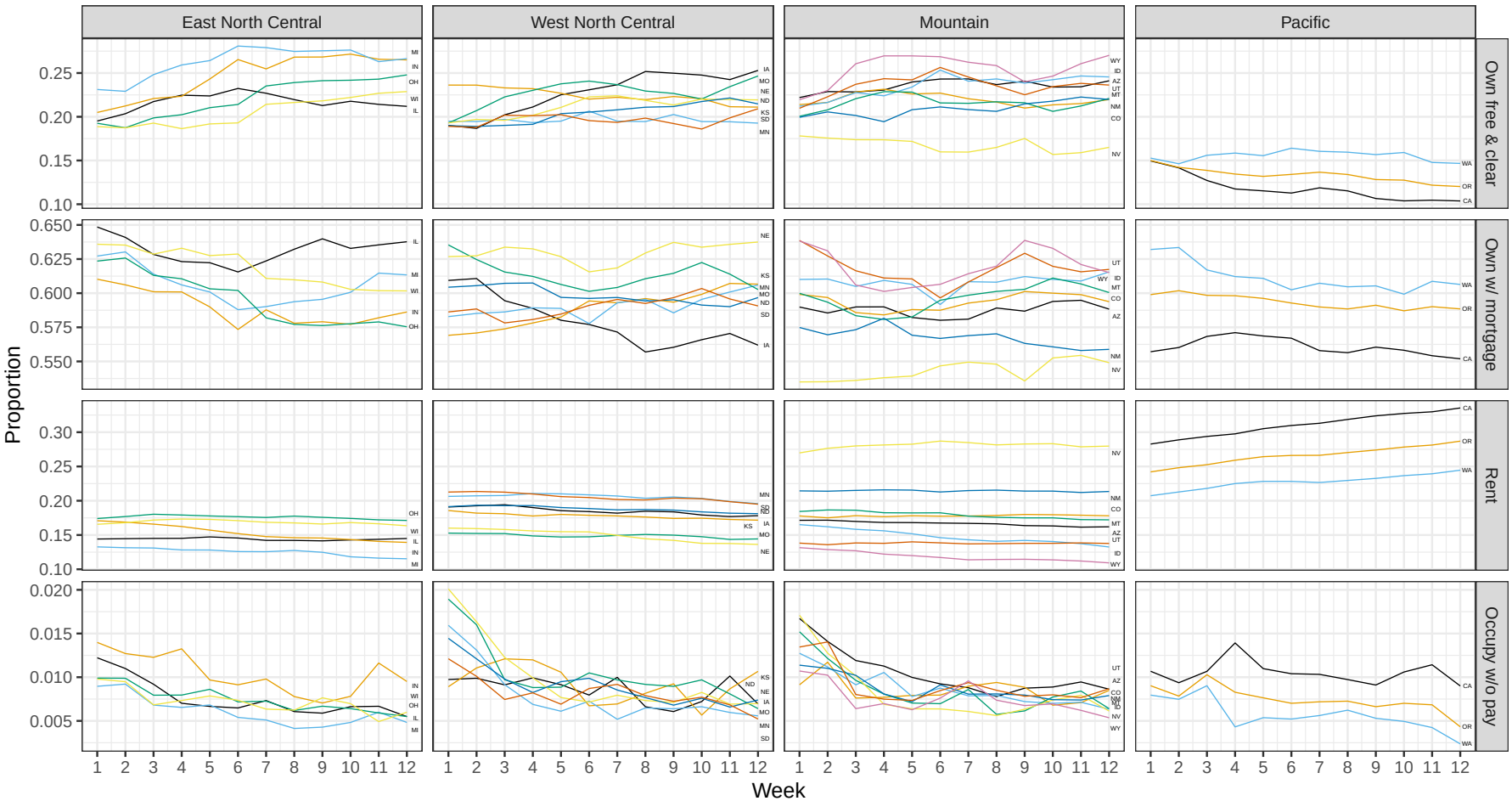
Setup

Fit our nominal model to the full HPS Phase 1 data

- Housing tenure type as response value
- Same covariates as before
- Use population-level covariates from Census Population Estimates program to generate posterior predictions

Data analysis

Estimated housing tenure, Asian males aged 45 to 50 by Census div.



Data analysis

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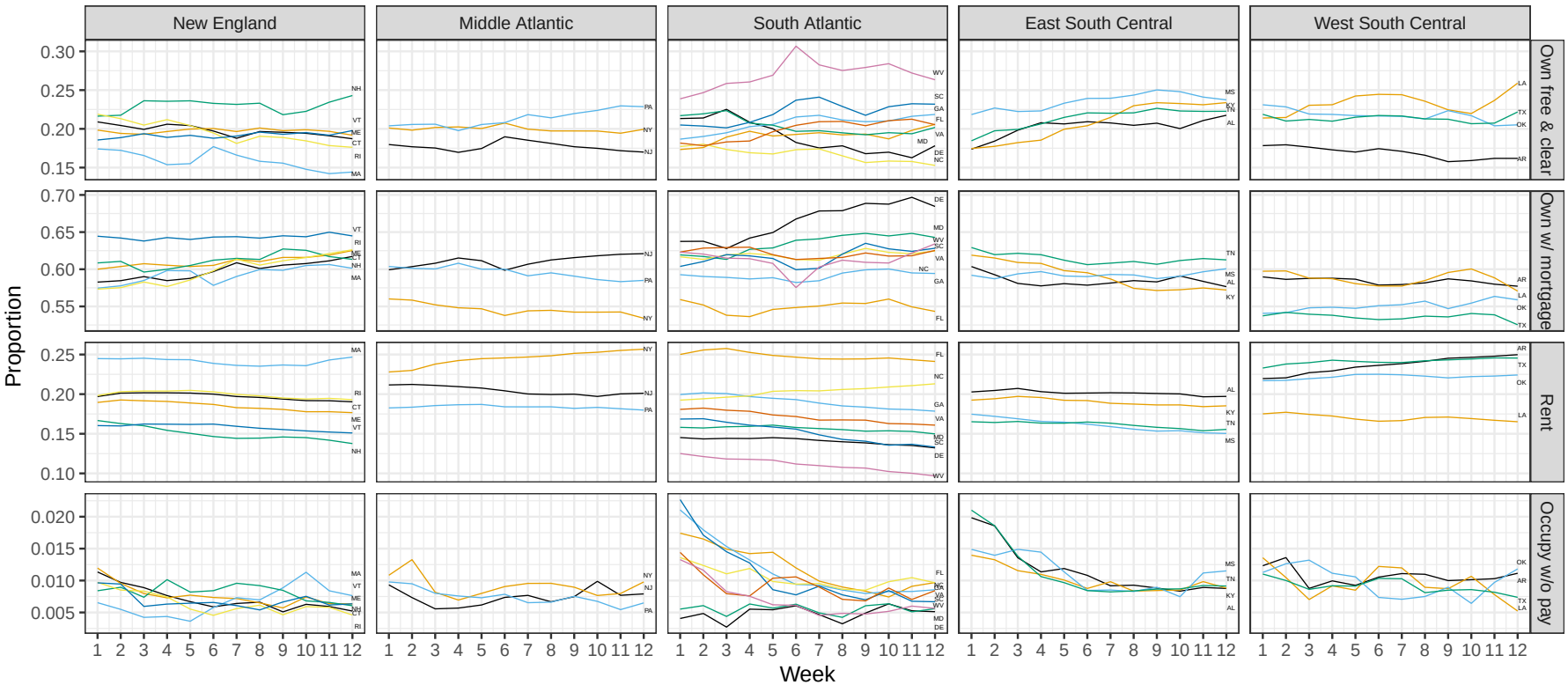

Summary

We propose unit-level models for nominal and ordinal data that

- account for longitudinal design
- account for informative sampling
- are computationally efficient

We illustrate their performance via

- an empirical simulation study
- and a full data analysis

Thank you!

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References

Albert, J. H. and Chib, S. (1993). "Bayesian Analysis of Binary and Polychotomous Response Data." *Journal of the American Statistical Association*, 88, 422, 669–679.

— (2001). "Sequential ordinal modeling with applications to survival data." *Biometrics*, 57, 3, 829–836.

Binder, D. A. (1983). "On the Variances of Asymptotically Normal Estimators from Complex Surveys." *International Statistical Review*, 51, 3, 279.

Fienberg, S. E. (1980). "The analysis of cross-classified categorical data." *Massachusetts Institute of Technology Press, Cambridge and London*.

Linderman, S., Johnson, M. J., and Adams, R. P. (2015). "Dependent Multinomial Models Made Easy: Stick-Breaking with the Polya-gamma Augmentation." In *Advances in Neural Information Processing Systems*, eds. C. Cortes, N. Lawrence, D. Lee, M. Sugiyama, and R. Garnett, vol. 28. Curran Associates, Inc.

McCullagh, P. (1980). "Regression models for ordinal data." *Journal of the Royal Statistical Society: Series B (Methodological)*, 42, 2, 109–127.

Parker, P. A., Holan, S. H., and Janicki, R. (2022). "Computationally Efficient Bayesian Unit-level Models for Non-Gaussian Data Under Informative Sampling with Application to Estimation of Health Insurance Coverage." *The Annals of Applied Statistics*, 16, 2, 887 – 904.

Parker, P. A., Janicki, R., and Holan, S. H. (2023). "A Comprehensive Overview of Unit-Level Modeling of Survey Data for Small Area Estimation Under Informative Sampling." *Journal of Survey Statistics and Methodology*.

Polson, N. G., Scott, J. G., and Windle, J. (2013). "Bayesian Inference for Logistic Models Using Pólya–Gamma Latent Variables." *Journal of the American Statistical Association*, 108, 504, 1339–1349.

Savitsky, T. D. and Toth, D. (2016). "Bayesian Estimation Under Informative Sampling." *Electronic Journal of Statistics*, 10, 1, 1677 – 1708.

Skinner, C. J. (1989). "Domain Means, Regression and Multivariate Analysis." In *Analysis of Complex Surveys*, eds. C. J. Skinner, D. Holt, and T. M. F. Smith, chap. 2, 59–84. Chichester: Wiley.

Tutz, G. (1990). "Sequential item response models with an ordered response." *British Journal of Mathematical and Statistical Psychology*, 43, 1, 39–55.

Vedensky, D., Parker, P. A., and Holan, S. H. (2023). "Bayesian Unit-level Models for Longitudinal Survey Data under Informative Sampling: An Analysis of Expected Job Loss Using the Household Pulse Survey."

Cumulative ordinal models

McCullagh (1980); Albert and Chib (1993)

Suppose

$$P(y_i \leq k|\boldsymbol{\gamma}, \boldsymbol{\mu}) = F(\gamma_k - \boldsymbol{\mu}_i)$$

which implies

$$P(y_i = k|\boldsymbol{\gamma}, \boldsymbol{\mu}) = F(\gamma_k - \boldsymbol{\mu}_i) - F(\gamma_{k-1} - \boldsymbol{\mu}_i)$$

or can be viewed as

$$y_i = k \text{ if } \gamma_{k-1} < \mu_i < \gamma_k.$$

Sequential ordinal models

Assuming

$$P(y_i = k | y_i \geq j, \beta, \eta, \gamma) = F(\gamma_k - \mu_i)$$

implies

$$P(y_i = k | \beta, \eta, \gamma) = F(\gamma_k - \mu_i) \prod_{j=1}^{k-1} (1 - F(\gamma_j - \mu_i)).$$

Alternatively, this can be viewed as saying

$$y_i = \begin{cases} 0 & \text{if } \mu_i \leq \gamma_1 \\ 1 & \text{if } \mu_i' > \gamma_1, \mu_i \leq \gamma_2 \\ \vdots & \vdots \\ K-1 & \text{if } \mu_i > \gamma_1, \mu_i > \gamma_2, \ldots, \mu_i > \gamma_{K-2}, \mu_i \leq \gamma_{K-1} \\ K & \text{if } \mu_i > \gamma_1, \mu_i > \gamma_2, \ldots, \mu_i > \gamma_{K-1} \end{cases}$$