Credible Distributions for Ranking of Entities

Gauri S. Datta

Department of Statistics, University of Georgia
Center for Statistical Research & Methodology, U.S. Census Bureau

A JSM memorial session for Dr. Ralph Folsom: August 6
Portland, Oregon, August 3-8, 2024
In collaboration with Yiren Hou and Abhyuday Mandal

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Outline

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Motivation

- Inference on overall ranking of a set of entities, such as chess players, subpopulations or hospitals, is an important problem.
- Accountability of public institutions involve in making quantitative comparisons between institutions in the areas of health and education.
- Inference of ranks based on point estimates of means does not account for the uncertainty in those estimates.
- Goldstein and Spiegelhalter (1996) and Klein, Wright and Wieczorek (2020) recognized treating estimated ranks without regard for uncertainty is problematic.
- KWW proposed a comprehensive frequentist solution. Following GS Bayesian approach, we propose a comprehensive Bayesian solution.

Main points of the talk

- Productions of accurate estimates of the means $\theta_1, \cdots, \theta_m$ for some characteristic for the subpopulations are usually the primary goals.
- It is also important to accurately identify subpopulations that are either at the upper or at the lower end in terms of their means. This goal requires accurately estimating the ranks of several or all the subpopulations.
- The importance of joint ranking of m subpopulations with unknown means of a common characteristic has been emphasized by KWW.

Importance of estimation of overall ranking

- KWW cautioned that from published point estimates of the means with no explicit ranking, practitioners frequently naively ascertain ranks of the subpopulations. Ranks determined this way are only point estimates, ignoring uncertainty.
- Even the best possible estimators of small area means are subject to error due to sampling variability. It is both imperative and a sound policy to evaluate uncertainty associated with the reported ranks based on reasonable estimators of means.
- KWW used ACS data to rank 50 U.S. states and DC, using mean commuting times of workers 16 years old and over, not working from home. They determined joint confidence set for the true rank vector from that of the true means.

Travel time data (upper half of Table 1 of KWW paper)

592 *M. Klein, T. Wright and J. Wieczorek*

Table 1. Mean travel time to work of workers 16 years old and over who did not work at home†

Rank	*Geographical area*	*Statistical significance?*	*Estimated mean (min)*	*Margin of error*
	USA		25.5	±0.1
51	Maryland		32.2	±0.2
50	New York		31.5	±0.2
49	New Jersey		30.5	±0.2
48	District of Columbia		30.1	±0.5
47	Illinois		28.2	±0.2
46	Massachusetts		28.0	±0.2
45	Virginia		27.7	±0.2
44	California		27.1	±0.1
44	Georgia		27.1	±0.3
42	New Hampshire		26.9	±0.5
41	Pennsylvania		25.9	±0.1
40	Florida		25.8	±0.2
39	Hawaii		25.7	±0.4
38	West Virginia		25.6	±0.5
37	Washington		25.5	±0.2
36	Delaware		25.3	±0.6
35	Connecticut		25.0	±0.3
34	Arizona		24.8	±0.2
34	Texas		24.8	±0.1
32	Colorado		24.5	±0.3
32	Louisiana		24.5	±0.2

Travel time data (lower half of Table 1 of KWW paper)

32	Louisiana		24.5	±0.2
30	Tennessee	‡	24.2	±0.2
29	Michigan	‡	24.1	±0.2
29	Nevada	‡	24.1	±0.4
27	*Alabama*	§	*23.9*	±*0.2*
27	Mississippi	‡	23.9	±0.4
25	South Carolina	‡	23.6	±0.3
24	Indiana		23.5	±0.2
23	Maine		23.4	±0.4
23	North Carolina		23.4	±0.2
23	Rhode Island	‡	23.4	±0.5
20	Missouri		23.1	±0.2
20	Ohio		23.1	±0.1
18	Minnesota		23.0	±0.2
17	Kentucky		22.9	±0.2
16	Oregon		22.5	±0.3
15	Vermont		21.9	±0.5
15	Wisconsin		21.9	±0.2
13	Utah		21.6	±0.3
12	New Mexico		21.4	±0.4
11	Arkansas		21.3	±0.4
10	Oklahoma		21.1	±0.2
9	Idaho		19.7	±0.4
8	Kansas		18.9	±0.3
7	Iowa		18.8	±0.2
6	Alaska		18.4	±0.5
5	Montana		18.2	±0.5
4	Nebraska		18.1	±0.3
4	Wyoming		18.1	±0.8
2	North Dakota		16.9	±0.6
2	South Dakota		16.9	±0.5

Early works

- Some of the early classical works on ranking and selection of population means are by Bechhofer (1954), Gupta (1956).
- A Bayesian approach to ranking several binomial populations: Bland and Bratcher (1968), Govindarajulu and Harvey (1974), Goel and Rubin (1977).
- Laird and Louis (1989) proposed an empirical Bayes (EB) approach, Morris and Christiansen (1994) proposed an approximate hierarchical Bayes (HB) approach.
- Aitkin and Longford (1986) and Laird and Louis (1989) used EB approach to ranking.
- Berger and Deely (1988) used an HB approach to ranking.

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Estimation of overall ranking

- Klein et al. (2020) considered overall ranking of the 50 US states and DC, based on θ_i, the mean commuting times of the workers not working from home, $i = 1, \cdots, m = 51$.
- True means θ_i's are estimated from the ACS data; $y_1, \cdots, y_m$ are the estimates from ACS
- $\check{r}_1, \cdots, \check{r}_m$ are the ranks of the true means
- $R_1, \cdots, R_m$ are the ranks of $y_1, \cdots, y_m$
- Estimates of the ranks $\check{r}_1, \cdots, \check{r}_m$ from $R_1, \cdots, R_m$, the ranks of the point estimates $y_1, \cdots, y_m$ ignore the sampling error in the y-estimates

Klein et al. joint confidence set for θ

- KWW assumed the model

$$Y_i|\theta \stackrel{ind}{\sim} N(\theta_i, D_i), \; i = 1, \cdots, m,$$

- Based on this, KWW created $(1-\alpha)$-level *joint confidence set* for θ based on confidence intervals

$$I_i = \left(Y_i - z_{1-\frac{\gamma}{2}} \sqrt{D_i}, \; Y_i - z_{\frac{\gamma}{2}} \sqrt{D_i} \right) \equiv (L_i, U_i),$$

  for $i = 1, \cdots, m$, for individual θ_i's. KWW determined γ from α by Bonferroni inequality or independence method.

- A joint $(1-\alpha)$-confidence set for θ is the Cartesian product of the intervals I_i, for $i = 1, \cdots, m$.

Klein et al. confidence solution for overall ranking

- From the joint confidence set $I_1 \times \cdots \times I_m$ of θ, KWW created confidence solution for overall ranking. For each $i \in \{1, 2, \cdots, m\}$, they define sets

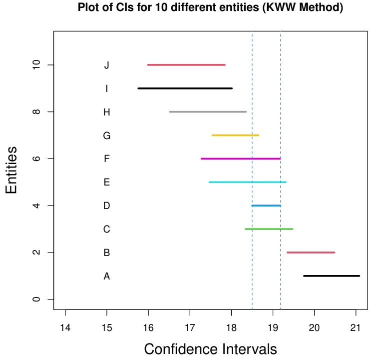
$$\begin{aligned} C_i &= \{1, 2, \cdots, m\} \setminus \{i\}, \\ \Lambda_{Li} &= \{j \in C_i : U_j \leq L_i\}, \\ \Lambda_{Ri} &= \{j \in C_i : U_i \leq L_j\}, \\ \Lambda_{Oi} &= \{j \in C_i : U_j > L_i \text{ and } U_i > L_j\} = C_i \setminus \{\Lambda_{Li} \cup \Lambda_{Ri}\} \end{aligned}$$

- Klein et al. (2020) show that the set of rank vectors

$$\left\{(r_1, \cdots, r_m) : r_i \in \{|\Lambda_{Li}| + 1, \cdots, |\Lambda_{Li}| + 1 + |\Lambda_{Oi}|\} \text{ for } i = 1, \cdots, m\right\},$$

is a joint confidence set for the overall rank vector $\check{r} = (\check{r}_1, \cdots, \check{r}_m)$ with coverage probability at least $(1 - \alpha)$.

Illustration of KWW Ranking



Consider D $(i=4)$. Then

$$\Lambda_{Li} = \{8,9,10\},\ \Lambda_{Ri} = \{1,2\},\ \Lambda_{Oi} = \{3,5,6,7\}$$
$$|\Lambda_{Li}| = 3,\ |\Lambda_{Oi}| = 4,\ r_i \in \{3+1,\cdots,3+4+1\} = \{4,5,6,7,8\}$$

Joint conf. set for ranking from jt CIs: Table 2 of KWW

Table 2 Joint confidence region for ranking based on joint confidence intervals (θ_ks): Bonferroni or independence (travel time to work data)†

$\hat{r}_k$	*State (k)*	$\hat{\theta}_k$	MOE_k	*Results for Bonferroni (10)*		*Results for independence (13)*	
				90% joint confidence intervals for θ_ks	*90% joint confidence region for ranking*	*90% joint confidence intervals for θ_ks*	*90% joint confidence region for ranking*
51	Maryland (MD)	32.2	0.2	(31.8, 32.6)	{50, 51}	(31.8, 32.6)	{50, 51}
50	New York (NY)	31.5	0.2	(31.1, 31.9)	{50, 51}	(31.1, 31.9)	{50, 51}
49	New Jersey (NJ)	30.5	0.2	(30.1, 30.9)	{48, 49}	(30.1, 30.9)	{48, 49}
48	District of Columbia (DC)	30.1	0.5	(29.2, 31.0)	{48, 49}	(29.2, 31.0)	{48, 49}
47	Illinois (IL)	28.2	0.2	(27.8, 28.6)	{45, 46, 47}	(27.8, 28.6)	{45, 46, 47}
46	Massachusetts (MA)	28.0	0.2	(27.6, 28.4)	{43, ..., 47}	(27.6, 28.4)	{43, ..., 47}
45	Virginia (VA)	27.7	0.2	(27.3, 28.1)	{43, ..., 47}	(27.3, 28.1)	{43, ..., 47}
44	California (CA)	27.1	0.1	(26.9, 27.3)	{42, 43, 44}	(26.9, 27.3)	{42, 43, 44}
44	Georgia (GA)	27.1	0.3	(26.5, 27.7)	{42, ..., 46}	(26.5, 27.7)	{42, ..., 46}
42	New Hampshire (NH)	26.9	0.5	(26.0, 27.8)	{37, ..., 46}	(26.0, 27.8)	{37, ..., 46}
41	Pennsylvania (PA)	25.9	0.1	(25.7, 26.1)	{36, ..., 42}	(25.7, 26.1)	{36, ..., 42}
40	Florida (FL)	25.8	0.2	(25.4, 26.2)	{35, ..., 42}	(25.4, 26.2)	{35, ..., 42}
39	Hawaii (HI)	25.7	0.4	(24.9, 26.5)	{32, ..., 42}	(25.0, 26.4)	{33, ..., 42}
38	West Virginia (WV)	25.6	0.5	(24.7, 26.5)	{30, ..., 42}	(24.7, 26.5)	{30, ..., 42}
37	Washington (WA)	25.5	0.2	(25.1, 25.9)	{34, ..., 41}	(25.1, 25.9)	{34, ..., 41}
36	Delaware (DE)	25.3	0.6	(24.2, 26.4)	{25, ..., 42}	(24.2, 26.4)	{25, ..., 42}
35	Connecticut (CT)	25.0	0.3	(24.4, 25.6)	{27, ..., 40}	(24.4, 25.6)	{27, ..., 40}
34	Arizona (AZ)	24.8	0.2	(24.4, 25.2)	{27, ..., 39}	(24.4, 25.2)	{27, ..., 39}
34	Texas (TX)	24.8	0.1	(24.6, 25.0)	{29, ..., 38}	(24.6, 25.0)	{30, ..., 37}
32	Colorado (CO)	24.5	0.3	(23.9, 25.1)	{23, ..., 38}	(23.9, 25.1)	{23, ..., 38}
32	Louisiana (LA)	24.5	0.2	(24.1, 24.9)	{23, ..., 37}	(24.1, 24.9)	{24, ..., 37}

Joint conf. set for ranking from jt CIs: Table 2 of KWW

32	Louisiana (LA)	24.5	0.2	(24.1, 24.9)	{23, ..., 37}	(24.1, 24.9)	{24, ..., 37}
30	Tennessee (TN)	24.2	0.2	(23.8, 24.6)	{22, ..., 35}	(23.8, 24.6)	{22, ..., 35}
29	Michigan (MI)	24.1	0.2	(23.7, 24.5)	{21, ..., 35}	(23.7, 24.5)	{21, ..., 35}
29	Nevada (NV)	24.1	0.4	(23.3, 24.9)	{19, ..., 37}	(23.4, 24.8)	{20, ..., 37}
27	Alabama (AL)	23.9	0.2	(23.5, 24.3)	{21, ..., 33}	(23.5, 24.3)	{21, ..., 33}
27	Mississippi (MS)	23.9	0.4	(23.1, 24.7)	{17, ..., 36}	(23.2, 24.6)	{17, ..., 35}
25	South Carolina (SC)	23.6	0.3	(23.0, 24.2)	{16, ..., 32}	(23.0, 24.2)	{16, ..., 32}
24	Indiana (IN)	23.5	0.2	(23.1, 23.9)	{17, ..., 30}	(23.1, 23.9)	{17, ..., 30}
23	Maine (ME)	23.4	0.4	(22.6, 24.2)	{15, ..., 32}	(22.7, 24.1)	{15, ..., 31}
23	North Carolina (NC)	23.4	0.2	(23.0, 23.8)	{16, ..., 29}	(23.0, 23.8)	{16, ..., 29}
23	Rhode Island (RI)	23.4	0.5	(22.5, 24.3)	{15, ..., 33}	(22.5, 24.3)	{15, ..., 33}
20	Missouri (MO)	23.1	0.2	(22.7, 23.5)	{15, ..., 27}	(22.7, 23.5)	{15, ..., 27}
20	Ohio (OH)	23.1	0.1	(22.9, 23.3)	{16, ..., 26}	(22.9, 23.3)	{16, ..., 26}
18	Minnesota (MN)	23.0	0.2	(22.6, 23.4)	{15, ..., 27}	(22.6, 23.4)	{15, ..., 26}
17	Kentucky (KY)	22.9	0.2	(22.5, 23.3)	{15, ..., 26}	(22.5, 23.3)	{15, ..., 26}
16	Oregon (OR)	22.5	0.3	(21.9, 23.1)	{11, ..., 24}	(21.9, 23.1)	{11, ..., 24}
15	Vermont (VI)	21.9	0.5	(21.0, 22.8)	{10, ..., 21}	(21.0, 22.8)	{10, ..., 21}
15	Wisconsin (WI)	21.9	0.2	(21.5, 22.3)	{11, ..., 16}	(21.5, 22.3)	{11, ..., 16}
13	Utah (UT)	21.6	0.3	(21.0, 22.2)	{10, ..., 16}	(21.0, 22.2)	{10, ..., 16}
12	New Mexico (NM)	21.4	0.4	(20.6, 22.2)	{10, ..., 16}	(20.7, 22.1)	{10, ..., 16}
11	Arkansas (AR)	21.3	0.4	(20.5, 22.1)	{10, ..., 16}	(20.6, 22.0)	{10, ..., 16}
10	Oklahoma (OK)	21.1	0.2	(20.7, 21.5)	{10, ..., 14}	(20.7, 21.5)	{10, ..., 14}
9	Idaho (ID)	19.7	0.4	(18.9, 20.5)	{4, ..., 9}	(19.0, 20.4)	{4, ..., 9}
8	Kansas (KS)	18.9	0.3	(18.3, 19.5)	{3, ..., 9}	(18.3, 19.5)	{3, ..., 9}
7	Iowa (IA)	18.8	0.2	(18.4, 19.2)	{3, ..., 9}	(18.4, 19.2)	{3, ..., 9}
6	Alaska (AK)	18.4	0.5	(17.5, 19.3)	{1, ..., 9}	(17.5, 19.3)	{1, ..., 9}
5	Montana (MT)	18.2	0.5	(17.3, 19.1)	{1, ..., 9}	(17.3, 19.1)	{1, ..., 9}
4	Nebraska (NE)	18.1	0.3	(17.5, 18.7)	{1, ..., 8}	(17.5, 18.7)	{1, ..., 8}
4	Wyoming (WY)	18.1	0.8	(16.6, 19.6)	{1, ..., 9}	(16.6, 19.6)	{1, ..., 9}
2	North Dakota (ND)	16.9	0.6	(15.8, 18.0)	{1, ..., 6}	(15.8, 18.0)	{1, ..., 6}
2	South Dakota (SD)	16.9	0.5	(16.0, 17.8)	{1, ..., 6}	(16.0, 17.8)	{1, ..., 6}

†Source: based on 2011 1-year ACS, ranking table R0801.

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Notion of a credible distribution of the overall ranking

- We construct Bayesian credible distributions of overall ranking by a sampling-based approach by drawing samples from the posterior distribution of θ.
- We generate a large sample $\mathcal{F}_M$, of size S, of $\theta_1, \cdots, \theta_m$ values from their joint posterior pdf, $\pi_M(\theta|y)$, derived under a model M.
- We empirically choose a suitable subsample of size $\approx S \times (1-\alpha)$ from these S samples; denote this set by $\mathcal{F}_{M,\alpha,y}$. This subsample corresponds to an appropriate credible set of θ.
- The set $\mathcal{F}_{M,\alpha,y}$ has empirical posterior probability $(1-\alpha)$. We rank each of the chosen θ and create a credible distribution for the true rank vector $\check{r}_1, \cdots, \check{r}_m$

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Two Bayesian models

An unstructured Bayesian model:

(I) $Y_i|\theta_1,\cdots,\theta_m \overset{ind}{\sim} N(\theta_i, D_i), i=1,\cdots,m,$

(II) $\pi(\theta_1,\cdots,\theta_m)=1$ for $-\infty<\theta_1,\cdots,\theta_m<\infty$. The joint posterior pdf of θ is a known MVN.

Fay-Herriot model: a class of hierarchical Bayesian models:

(I) $Y_i|\theta_1,\cdots,\theta_m \overset{ind}{\sim} N(\theta_i, D_i), i=1,\cdots,m,$

(II) Conditional on model parameters β, A, subpopulation means θ_i's are independently distributed, given by $\theta_i|\beta,A \overset{ind}{\sim} N(x_i^T\beta, A), i=1,\cdots,m,$

(III) $\pi(\beta,A)=1/\sqrt{\bar{D}+A}$. Posterior distribution of θ can be easily sampled by MCMC or by non-Markovian method.

Credible set for θ: A Cartesian credible set

- From the sample in $\mathcal{F}_M$, we take the ith component of each vector to create a sample $\mathcal{F}_i$ for θ_i. For a suitable κ, we determine the $(\kappa/2)$th and $(1-\kappa/2)$th quantiles, a_i and b_i, of the set $\mathcal{F}_i$. Define now

$$\begin{aligned} \mathcal{S}_i &= \{s : a_i \leq \theta_i^{(s)} \leq b_i, s = 1, 2, \cdots S\} \\ \mathcal{S}_J &= \cap_{i=1}^{m} \mathcal{S}_i \end{aligned}$$

  Let $K_J = |\mathcal{S}_J|$. For any $\alpha \in (0,1)$, we determine κ in such a way that K_J is the integer closest to $S \times (1-\alpha)$.

- The selected set

$$\{\theta^{(s)}, s \in \mathcal{S}_J\}$$

  is an approximate representation of a $(1-\alpha)$ joint credible set for θ. We use these θ values to determine a rank distribution, described below.

A nearly optimal credible set for θ: an elliptical set

- An optimal method to create credible set for θ is the HPD method.
  - For the UB model, HPD credible set is elliptical.
  - For the HB model, an HPD set is approximately elliptical.
- The HPD credible set is centered at

$$\hat{\theta} = \frac{1}{S}\sum_{s=1}^{S} \theta^{(s)}$$

and the elliptical shape is determined by the dispersion matrix

$$V = \frac{1}{S}\sum_{s=1}^{S} \{\theta^{(s)} - \hat{\theta}\}\{\theta^{(s)} - \hat{\theta}\}^T.$$

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A nearly optimal credible set for θ: an elliptical set

- Create Mahalanobis distances
$$d_B^{(s)} = M\left(\theta^{(s)}, \hat{\theta}, V\right), \text{ for } s = 1, \cdots, S.$$
Find c, the $(1-\alpha)$-quantile of these distances, and create an empirical credible set $\mathcal{F}_{M,\alpha,y}$ for θ with $\{s \in \mathcal{F}_M : d_B^{(s)} \leq c\}$.
- The selected set $\mathcal{F}_{M,\alpha,y}$ is an approximate representation of a $(1-\alpha)$ joint HPD credible set for θ.
- Again, we use these θ values to determine a rank distribution below.

Joint credible distribution of overall ranking

- For each sample in $\mathcal{F}_{M,\alpha,y}$, we create a two-way $m \times m$ table, columns as the populations or subjects to be ranked and rows as the ranks of the components of θ, explained now. For each sample such as $\theta^{(s_1)}$, we start with an $m \times m$ null matrix.
- If the ith component of $\theta^{(s_1)}$ does not tie with any other component and has rank j_i, we put a 1 in the j_ith row of the ith column. If two components k and l tie for ranks j and $j+1$, we replace elements $(k,j),(k,j+1),(l,j),(l,j+1)$ by $1/2$.
- For all K elements in $\mathcal{F}_{M,\alpha,y}$, we complete K tables, and for some weight $w(\theta)$ we take a weighted average of these tables, where the weights sum to 1. This produces a credible distribution of overall ranking. We denote the distribution of $\check{r}_i$ by $\pi_{\check{r}_i,M,T,y}$.

Outline

Ranking baseball players: An example by Efron and Morris

- Efron and Morris (1975) considered batting averages of 18 major league baseball batters from their first 45 at bats in the 1970 baseball season to predict their performances in the remainder of that season.
- A reasonable representative value for θ_i was the player's average (ω_i), known, in the remainder of the season after his first 45 at bats.
- The sample proportion of hits Y_i from first 45 at bats is approximately normal with mean θ_i and estimated variance $D_i = Y_i(1 - Y_i)/45$.

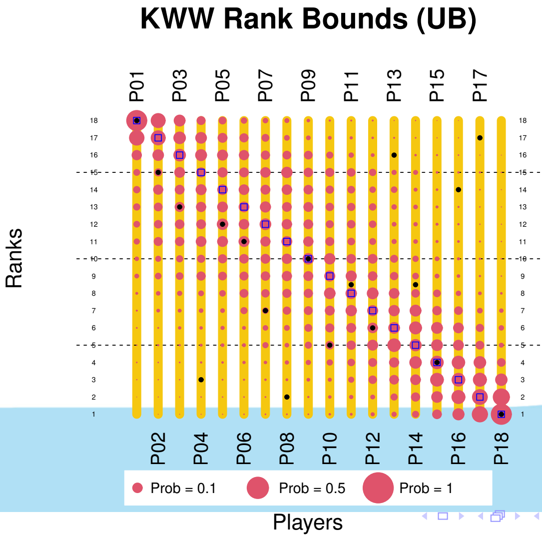
KWW confidence set of overall ranking of players

- From equation (9) of Klein et al. (2020) a 90% joint confidence region for the rank vector $\check{r}$ is given by

$$\{(\check{r}_1, \cdots, \check{r}_{18}) : \check{r}_k \in \{1, 2, \cdots, 18\} \text{ for } k = 1, \cdots, 18\}.$$

  - This confidence set for the rank vector is depicted in the Figure next slide.
  - Possible ranks for each player in this set are shown by a yellow line segment stretching from 1 to 18.
  - According to this confidence set, any player can rank from 1 to 18, and any particular ranks can be held by any of these 18 players.
- The credible distribution is pictorially presented by overlaying on the figure of the confidence set solid red circles with area of a circle is proportional to the probability it is representing.

Conf sets and credible distrns (Baseball)



Comparison of confidence sets and credible distributions

- ω_i: Batting average from the remainder of the season after first 45 at bats for player i.
- ξ_i: A surrogate of true rank based on ω_i.
- Compute for player $i = 1, \cdots, 18$,

$$\varepsilon_{i,HB,C} = E^{\pi_{\check{r}_i,HB,C,y}}\left(|\check{r}_i - \xi_i| \Big| y\right), \qquad \varepsilon_{i,HB,E} = E^{\pi_{\check{r}_i,HB,E,y}}\left(|\check{r}_i - \xi_i| \Big| y\right),$$

$$\varepsilon_{i,UB,C} = E^{\pi_{\check{r}_i,UB,C,y}}\left(|\check{r}_i - \xi_i| \Big| y\right), \qquad \varepsilon_{i,UB,E} = E^{\pi_{\check{r}_i,UB,E,y}}\left(|\check{r}_i - \xi_i| \Big| y\right),$$

$$\varepsilon_{i,KWW} = \frac{1}{|\Lambda_{Oi}|+1} \sum_{j=|\Lambda_{Li}|+1}^{|\Lambda_{Li}|+|\Lambda_{Oi}|+1} |j - \xi_i|.$$

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Application to 1970 Batting averages of 18 Major League players

Players	y_i	ω_i	R_i	ξ_i	$\varepsilon_{i,HB,C}$	$\varepsilon_{i,HB,E}$	$\varepsilon_{i,UB,C}$	$\varepsilon_{i,UB,E}$	$\varepsilon_{i,KWW}$
1. Clemente	.400	.346	18	18	5.90	5.90	1.18	1.18	8.50
2. F. Robinson	.378	.298	17	15	4.39	4.39	1.87	1.87	6.17
3. F. Howard	.356	.276	16	13	4.24	21.0	2.73	2.73	5.17
4. Johnstone	.333	.222	15	3	8.11	8.11	11.40	11.39	**6.83**
5. Berry	.311	.273	13.5	12	4.33	4.33	2.44	2.44	4.83
6. Spencer	.311	.270	13.5	11	4.25	4.25	2.96	2.96	4.61
7. Kessinger	.289	.263	12	7	4.96	4.96	4.92	4.92	4.83
8. L. Alvarado	.267	.210	11	2	7.66	7.66	8.13	8.13	**7.61**
9. Santo	.244	.269	9.5	10	4.36	4.34	3.09	3.09	4.50
10. Swoboda	.244	.230	9.5	5	5.41	5.41	3.98	3.98	5.61
11. Unser	.222	.264	6	8.5	4.26	4.26	3.09	3.08	4.56
12. Williams	.222	.256	6	6	4.40	4.40	2.64	2.64	5.17
13. Scott	.222	.303	6	16	7.64	7.64	9.36	9.36	**6.83**
14. Petrocelli	.222	.264	6	8.5	4.25	4.25	3.33	3.33	4.56
15. E. Rodriguez	.222	.226	6	4	5.38	5.38	3.46	3.46	6.17
16. Campenaris	.200	.285	3	14	6.64	6.64	8.97	8.97	**5.61**
17. Munson	.178	.316	2	17	9.54	9.54	13.57	13.56	**7.61**
18. Alvis	.156	.200	1	1	5.46	5.46	1.43	1.43	8.50
Total abs. dev.					101.16	101.16	88.56	88.56	107.64
$10^{13} \times$ Vol.					51.38	1.32	93756	1757	85722
Average length					0.236	0.193	0.359	0.288	0.357

Outline

Ranking of US states based on commuting times

- Recall that, in a pioneering article, Klein et al. (2020) applied their frequentist approach to rank fifty states of the U.S. and DC by mean commuting times of workers sixteen or older and not working from home. They used survey data collected from the ACS.
- Let us revisit their Table 2 next.

Joint conf. set for ranking from jt CIs: Table 2 of KWW

Table 2 Joint confidence region for ranking based on joint confidence intervals (θ_ks): Bonferroni or independence (travel time to work data)†

$\hat{r}_k$	*State (k)*	$\hat{\theta}_k$	MOE_k	*Results for Bonferroni (10)*		*Results for independence (13)*	
				90% joint confidence intervals for θ_ks	*90% joint confidence region for ranking*	*90% joint confidence intervals for θ_ks*	*90% joint confidence region for ranking*
51	Maryland (MD)	32.2	0.2	(31.8, 32.6)	{50, 51}	(31.8, 32.6)	{50, 51}
50	New York (NY)	31.5	0.2	(31.1, 31.9)	{50, 51}	(31.1, 31.9)	{50, 51}
49	New Jersey (NJ)	30.5	0.2	(30.1, 30.9)	{48, 49}	(30.1, 30.9)	{48, 49}
48	District of Columbia (DC)	30.1	0.5	(29.2, 31.0)	{48, 49}	(29.2, 31.0)	{48, 49}
47	Illinois (IL)	28.2	0.2	(27.8, 28.6)	{45, 46, 47}	(27.8, 28.6)	{45, 46, 47}
46	Massachusetts (MA)	28.0	0.2	(27.6, 28.4)	{43, ..., 47}	(27.6, 28.4)	{43, ..., 47}
45	Virginia (VA)	27.7	0.2	(27.3, 28.1)	{43, ..., 47}	(27.3, 28.1)	{43, ..., 47}
44	California (CA)	27.1	0.1	(26.9, 27.3)	{42, 43, 44}	(26.9, 27.3)	{42, 43, 44}
44	Georgia (GA)	27.1	0.3	(26.5, 27.7)	{42, ..., 46}	(26.5, 27.7)	{42, ..., 46}
42	New Hampshire (NH)	26.9	0.5	(26.0, 27.8)	{37, ..., 46}	(26.0, 27.8)	{37, ..., 46}
41	Pennsylvania (PA)	25.9	0.1	(25.7, 26.1)	{36, ..., 42}	(25.7, 26.1)	{36, ..., 42}
40	Florida (FL)	25.8	0.2	(25.4, 26.2)	{35, ..., 42}	(25.4, 26.2)	{35, ..., 42}
39	Hawaii (HI)	25.7	0.4	(24.9, 26.5)	{32, ..., 42}	(25.0, 26.4)	{33, ..., 42}
38	West Virginia (WV)	25.6	0.5	(24.7, 26.5)	{30, ..., 42}	(24.7, 26.5)	{30, ..., 42}
37	Washington (WA)	25.5	0.2	(25.1, 25.9)	{34, ..., 41}	(25.1, 25.9)	{34, ..., 41}
36	Delaware (DE)	25.3	0.6	(24.2, 26.4)	{25, ..., 42}	(24.2, 26.4)	{25, ..., 42}
35	Connecticut (CT)	25.0	0.3	(24.4, 25.6)	{27, ..., 40}	(24.4, 25.6)	{27, ..., 40}
34	Arizona (AZ)	24.8	0.2	(24.4, 25.2)	{27, ..., 39}	(24.4, 25.2)	{27, ..., 39}
34	Texas (TX)	24.8	0.1	(24.6, 25.0)	{29, ..., 38}	(24.6, 25.0)	{30, ..., 37}
32	Colorado (CO)	24.5	0.3	(23.9, 25.1)	{23, ..., 38}	(23.9, 25.1)	{23, ..., 38}
32	Louisiana (LA)	24.5	0.2	(24.1, 24.9)	{23, ..., 37}	(24.1, 24.9)	{24, ..., 37}

Joint conf. set for ranking from jt CIs: Table 2 of KWW

32	Louisiana (LA)	24.5	0.2	(24.1, 24.9)	{23, ..., 37}	(24.1, 24.9)	{24, ..., 37}
30	Tennessee (TN)	24.2	0.2	(23.8, 24.6)	{22, ..., 35}	(23.8, 24.6)	{22, ..., 35}
29	Michigan (MI)	24.1	0.2	(23.7, 24.5)	{21, ..., 35}	(23.7, 24.5)	{21, ..., 35}
29	Nevada (NV)	24.1	0.4	(23.3, 24.9)	{19, ..., 37}	(23.4, 24.8)	{20, ..., 37}
27	Alabama (AL)	23.9	0.2	(23.5, 24.3)	{21, ..., 33}	(23.5, 24.3)	{21, ..., 33}
27	Mississippi (MS)	23.9	0.4	(23.1, 24.7)	{17, ..., 36}	(23.2, 24.6)	{17, ..., 35}
25	South Carolina (SC)	23.6	0.3	(23.0, 24.2)	{16, ..., 32}	(23.0, 24.2)	{16, ..., 32}
24	Indiana (IN)	23.5	0.2	(23.1, 23.9)	{17, ..., 30}	(23.1, 23.9)	{17, ..., 30}
23	Maine (ME)	23.4	0.4	(22.6, 24.2)	{15, ..., 32}	(22.7, 24.1)	{15, ..., 31}
23	North Carolina (NC)	23.4	0.2	(23.0, 23.8)	{16, ..., 29}	(23.0, 23.8)	{16, ..., 29}
23	Rhode Island (RI)	23.4	0.5	(22.5, 24.3)	{15, ..., 33}	(22.5, 24.3)	{15, ..., 33}
20	Missouri (MO)	23.1	0.2	(22.7, 23.5)	{15, ..., 27}	(22.7, 23.5)	{15, ..., 27}
20	Ohio (OH)	23.1	0.1	(22.9, 23.3)	{16, ..., 26}	(22.9, 23.3)	{16, ..., 26}
18	Minnesota (MN)	23.0	0.2	(22.6, 23.4)	{15, ..., 27}	(22.6, 23.4)	{15, ..., 26}
17	Kentucky (KY)	22.9	0.2	(22.5, 23.3)	{15, ..., 26}	(22.5, 23.3)	{15, ..., 26}
16	Oregon (OR)	22.5	0.3	(21.9, 23.1)	{11, ..., 24}	(21.9, 23.1)	{11, ..., 24}
15	Vermont (VI)	21.9	0.5	(21.0, 22.8)	{10, ..., 21}	(21.0, 22.8)	{10, ..., 21}
15	Wisconsin (WI)	21.9	0.2	(21.5, 22.3)	{11, ..., 16}	(21.5, 22.3)	{11, ..., 16}
13	Utah (UT)	21.6	0.3	(21.0, 22.2)	{10, ..., 16}	(21.0, 22.2)	{10, ..., 16}
12	New Mexico (NM)	21.4	0.4	(20.6, 22.2)	{10, ..., 16}	(20.7, 22.1)	{10, ..., 16}
11	Arkansas (AR)	21.3	0.4	(20.5, 22.1)	{10, ..., 16}	(20.6, 22.0)	{10, ..., 16}
10	Oklahoma (OK)	21.1	0.2	(20.7, 21.5)	{10, ..., 14}	(20.7, 21.5)	{10, ..., 14}
9	Idaho (ID)	19.7	0.4	(18.9, 20.5)	{4, ..., 9}	(19.0, 20.4)	{4, ..., 9}
8	Kansas (KS)	18.9	0.3	(18.3, 19.5)	{3, ..., 9}	(18.3, 19.5)	{3, ..., 9}
7	Iowa (IA)	18.8	0.2	(18.4, 19.2)	{3, ..., 9}	(18.4, 19.2)	{3, ..., 9}
6	Alaska (AK)	18.4	0.5	(17.5, 19.3)	{1, ..., 9}	(17.5, 19.3)	{1, ..., 9}
5	Montana (MT)	18.2	0.5	(17.3, 19.1)	{1, ..., 9}	(17.3, 19.1)	{1, ..., 9}
4	Nebraska (NE)	18.1	0.3	(17.5, 18.7)	{1, ..., 8}	(17.5, 18.7)	{1, ..., 8}
4	Wyoming (WY)	18.1	0.8	(16.6, 19.6)	{1, ..., 9}	(16.6, 19.6)	{1, ..., 9}
2	North Dakota (ND)	16.9	0.6	(15.8, 18.0)	{1, ..., 6}	(15.8, 18.0)	{1, ..., 6}
2	South Dakota (SD)	16.9	0.5	(16.0, 17.8)	{1, ..., 6}	(16.0, 17.8)	{1, ..., 6}

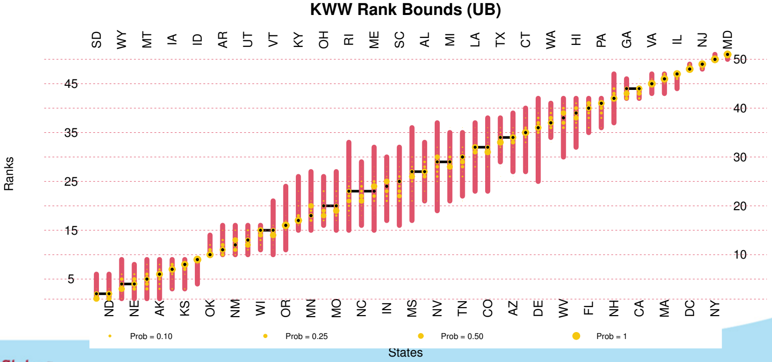
†Source: based on 2011 1-year ACS, ranking table R0801.

Ranking of US states based on commuting times

- In the figure below, we recreated Figure 1 of Klein et al. (2020) that depicted the frequentist solution of the confidence region for ranking.
- From the figure, for example, the possible ranks from this solution for the state ID are $4-9$.
- On the other hand, the states which can hold the rank 9 are the states WY, AK, MT, IA, KS and ID (the pink line segments for these states intersect the horizontal line for ranks = 9).

KWW2020 Figure 1

Visualization of the 90% joint CR for travel time ranking given by Figure 1 of Klein et al. (2020) with an overlapping credible distribution from UB method

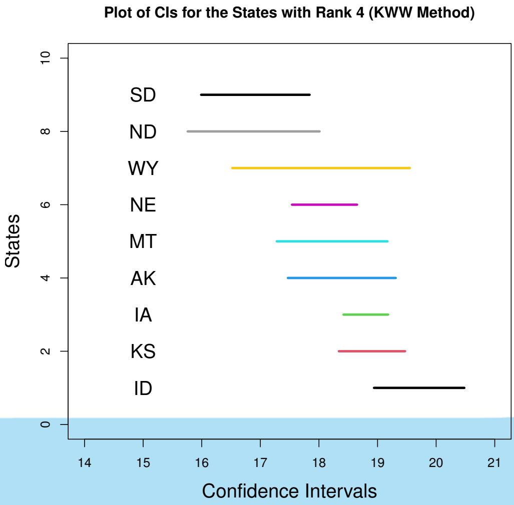


Credible distribution from Cartesian credible intervals

- On the last figure we overlaid credible distribution based on a C. credible set from the UB model. Probabilities from credible distribution are shown by yellow circles, bigger circles for larger probabilities.
- To interpret probabilities depicted in the figure, we focus specifically on its 4th row and 4th column. From the 4th column of this figure we find that the state of NE can have ranks 3–6 with respective probabilities about $0.2, 0.5, 0.2$, and 0.1.
- The 4th column in the figure shows the rank set for NE based on Klein et al. (2020) solution. It shows that in addition to the four ranks 3 to 6, NE can also rank 1, 2, 7 and 8.
- For ID, 9th column shows the KWW rank set $\{4, \cdots, 9\}$. Rank 9 on the 9th row is captured by one of six states: WY, MT, IA, AK, KS, ID. However, credible probabilities for ID and for rank 9 are 1.

United States Census Bureau

CI’s for means of certain contender states (Rank = 4)



Nine states likely at rank 4 (Q4 of Table 2 from KWW)

10	Oklahoma (OK)	21.1	0.2	(20.7, 21.5)	{10, . . . , 14}	(20.7, 21.5)	{10, . . . , 14}
9	Idaho (ID)	19.7	0.4	(18.9, 20.5)	{4, . . . , 9}	(19.0, 20.4)	{4, . . . , 9}
8	Kansas (KS)	18.9	0.3	(18.3, 19.5)	{3, . . . , 9}	(18.3, 19.5)	{3, . . . , 9}
7	Iowa (IA)	18.8	0.2	(18.4, 19.2)	{3, . . . , 9}	(18.4, 19.2)	{3, . . . , 9}
6	Alaska (AK)	18.4	0.5	(17.5, 19.3)	{1, . . . , 9}	(17.5, 19.3)	{1, . . . , 9}
5	Montana (MT)	18.2	0.5	(17.3, 19.1)	{1, . . . , 9}	(17.3, 19.1)	{1, . . . , 9}
4	Nebraska (NE)	18.1	0.3	(17.5, 18.7)	{1, . . . , 8}	(17.5, 18.7)	{1, . . . , 8}
4	Wyoming (WY)	18.1	0.8	(16.6, 19.6)	{1, . . . , 9}	(16.6, 19.6)	{1, . . . , 9}
2	North Dakota (ND)	16.9	0.6	(15.8, 18.0)	{1, . . . , 6}	(15.8, 18.0)	{1, . . . , 6}
2	South Dakota (SD)	16.9	0.5	(16.0, 17.8)	{1, . . . , 6}	(16.0, 17.8)	{1, . . . , 6}

Credible Distrns of Ranks vs. Conf. Sets (a portion)

KWW example: Comparison of credible distributions of ranks (UB-C) for states with the KWW jt. conf. sets of ranks. States considered are contenders of rank 4 by KWW solution.

Rank	**ID**	**KS**	**IA**	**AK**	**MT**	NE	**WY**	ND	SD
1							0.01	37.37	62.62
2							0.15	62.58	37.27
3				1.76	16.40	20.86	60.81	0.06	0.11
4				3.91	22.53	55.41	18.15		
5		0.79	0.37	13.81	51.67	20.10	13.26		
6		1.75	4.33	76.16	9.03	3.64	5.10		
7		13.10	81.30	3.91	0.34		1.35		
8		84.35	14.00	0.46	0.03		1.16		
9	100								
KWW set	4-9	3-9	3-9	1-9	1-9	1-8	1-9	1-6	1-6
Mean	19.7	18.9	18.8	18.4	18.2	18.1	18.1	16.9	16.9
MOE	0.749	0.562	0.375	0.936	0.936	0.562	1.498	1.124	0.936

Outline

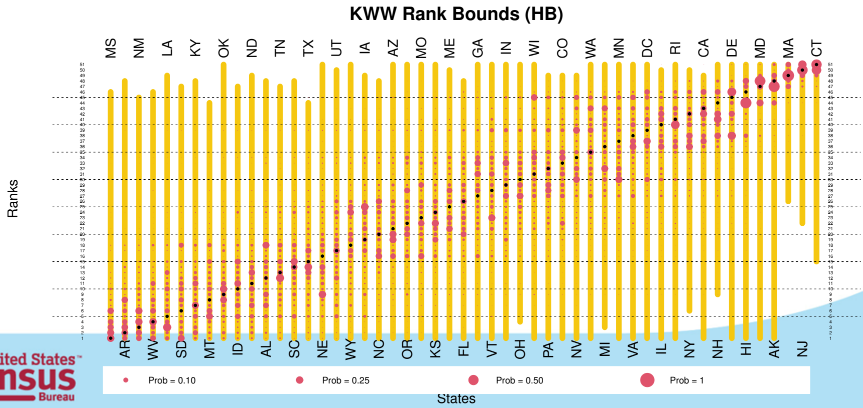
United States Census Bureau

Ranking of US states based on median incomes

- Inference on median incomes of the US states for the income year 1989 has served as a benchmark example to evaluate effectiveness of various small area estimation methods.
- Fay (1987) suggested using two covariates for Fay-Herriot model. x_{i1} is the ith state median income for 1979 from 1980 Census, and $x_{i2} = (\text{PCI}_{i,1989}/\text{PCI}_{i,1979})x_{i1}$, $i = 1, \ldots, m$, where $\text{PCI}_{i,1979}$ 1979 per capita income of the ith state.
- The 1990 census incomes for states were medians based on a large number of households from each state. These are "gold standard". This will be used to create "surrogate ranks" ξ_i for true ranks $\check{r}$.

A visualization of median income ranking

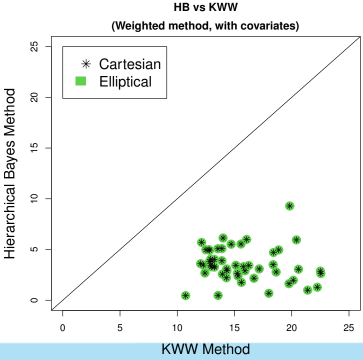
90% joint conf region for median income ranking by KWW with an overlapping credible distrn by HB method



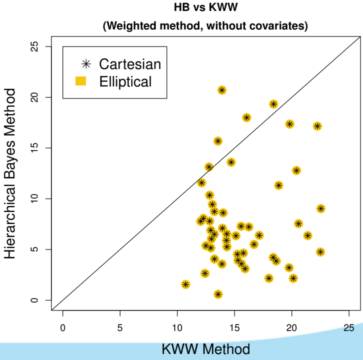
United States Census Bureau

Plots of expected or average absolute deviations

HB vs. KWW



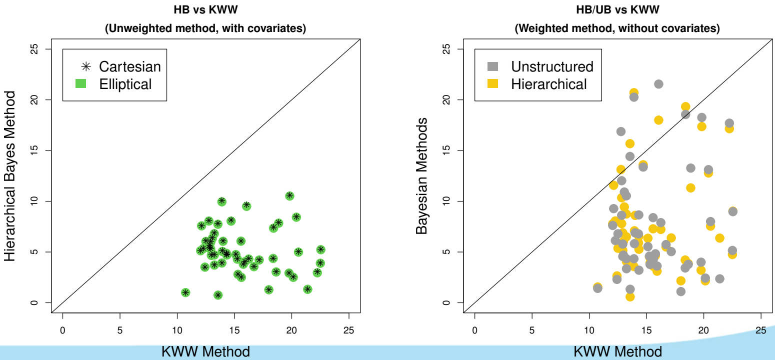
With Covariates



Covariates not used

Plots of expected or ave abs deviations

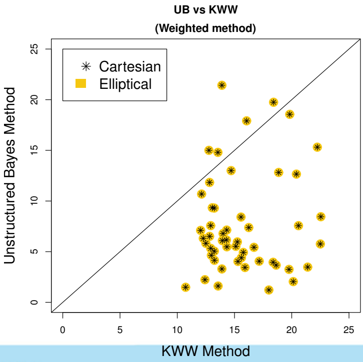
HB/UB vs. KWW

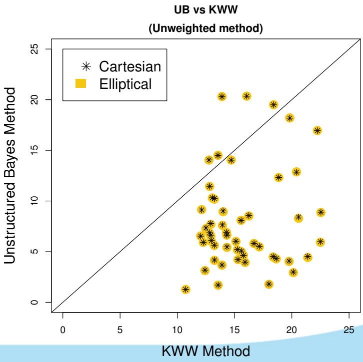


With Covariates | No covariates, elliptical

Plots of expected or average absolute deviations

UB-W vs. KWW and UB-UW vs. KWW





Covariates cannot be used

Covariates cannot be used

THANK YOU!!!

emails: gauri@uga.edu, gauri.datta@census.gov

References:

1. Klein, M., Wright, T. and Wieczorek, J. (2020). "A Joint Confidence Region for an Overall Ranking of Populations", *Journal of the Royal Statistical Society (Series C)*, 69, Part 3, 589–606.
2. Datta, G.; Hou, Y. and Mandal, A. (2024), "Credible Distributions of Overall Ranking of Entities", submitted. `https://arxiv.org/pdf/2401.01833.pdf`