Small Area Estimates for End-of-Season Agricultural Quantities

Andreea L. Erciulescu* Nathan B. Cruze † Balgobin Nandram‡

Abstract
The publication of official statistics at different levels of aggregation requires a benchmarking step. Difficulties arise when a benchmarking method needs to be applied to a triplet of related estimates, at multiple stages of aggregation. For ratios of totals, external benchmarking constraints for the triplet (numerator, denominator, ratio) are that the weighted sum of denominator/numerator/ratio estimates equals to a constant. The benchmarking weight applied to the ratio estimates is a function of the denominator estimates. For example, the United States Department of Agriculture's National Agricultural Statistics Service's county-level, end-of-season acreage, production and yield estimates need to aggregate to the corresponding agricultural statistics district-level estimates, that also need to aggregate to the corresponding prepublished, state-level values. Moreover, the definition of yield, as the ratio of production to harvested acreage, needs to hold at the county level, at the agricultural statistics district level and at the state level. We discuss different approaches of applying benchmarking constraints to a triplet (numerator, denominator, ratio), at multiple stages of aggregation, where estimators are constructed for two of the three quantities, the third being derived as a result. County-level and agricultural statistics district-level, end-of-season acreage, production and yield estimates are constructed and compared using the different methods. Results are illustrated for a subset of the sampled commodities and states, in the year 2014.

Key Words: Auxiliary Information, Bayes Estimates, Crop Estimates, End-of Season Yield, Multiple Stage Benchmarking, Official Statistics.

1. Introduction

Statistical agencies and policy makers require that the official statistics for nested levels are consistent. For the case where estimates for a large area, containing small areas, are published before the estimates for the small areas are constructed, it is desirable that the estimates for the small areas aggregate to the prepublished values. The process of adjusting the estimates, to satisfy external upper-level constraints, is known as external, top-down, benchmarking. The agreement between lower-level and upper-level estimates provides confidence for policy makers when utilizing official estimates. Also, the verified benchmarking constraint provides protection against an imperfect model, when the final estimates are produced using model-based methods; see Pfeffermann and Barnard (1991).

Estimation and benchmarking methods produce reliable and reproducible official estimates at nested levels. Benchmarking methods were developed for estimates of a single quantity of interest, at one level of aggregation, by a number of authors, see for example Pfeffermann and Barnard (1991), Wang, Fuller and Qu (2008), Nandram and Sayit (2014), Rao and Molina (2016). Ghosh and Steorts (2013) propose a two-stage benchmarking methodology, with extensions to multiple, *unrelated*, parameters of interest. In this paper, we consider estimation for the *triplet* (numerator, denominator, ratio), at two

*National Institute of Statistical Sciences and USDA National Agricultural Statistics Service, 1400 Independence Avenue, SW, Room 6412 B, Washington, DC 20250-2054. E-mail: alerciulescu@gmail.com

†USDA National Agricultural Statistics Service, 1400 Independence Avenue, SW, Room 6412 A, Washington, DC 20250-2054

‡Worcester Polytechnic Institute and USDA National Agricultural Statistics Service, Department of Mathematical Sciences, Stratton Hall, 100 Institute Road, Worcester, MA 01609

levels of aggregation. This is a difficult problem because the final triplet estimates need to satisfy identity constraints (ratio of numerator to denominator) as well as benchmarking constraints at multiple levels. We assume that Markov chain Monte Carlo (MCMC) iterates are available for the denominator total and for the ratio of two totals, as a result of independent model fitting. Estimates are produced using the MCMC iterates of the two quantities, the third (the numerator total) being produced as a result. Our goal is to discuss the application of the benchmarking adjustment to the triplet of estimates and its consequences to maintaining the ratio identity at different aggregation levels.

The motivation for this research is county-level and agricultural statistics district-level yield estimation at United States Department of Agriculture's (USDA's) National Agricultural Statistics Service (NASS), where yield is defined as the ratio of production to harvested acreage. NASS produces county-level and agricultural statistics district-level estimates (groups of neighboring counties within a state, hereafter denoted by ASDs) of acreage, production and yield that may contribute to the magnitude of payout in some agricultural programs; see Cruze et al. (2016). Currently, NASS publishes end-of-season county-level estimates produced using a top-down approach, for a unique year-state-commodity combination. National and state end-of-season crop estimates are produced using NASS's quarterly crops Acreage, Production and Stocks (APS) surveys, and published prior to county-level estimates. Agricultural statistics district-level estimates are then produced such that they aggregate to the state-level values. End-of-season crop estimates at the county level are produced using the APS surveys, and their supplement County Agricultural Production Survey (CAPS). The county-level estimates need to aggregate to the prepublished state-level values, while the intermediate agreements with the ASDs need to hold.

Because the current NASS estimation method uses expert opinion to combine multiple sources of data to produce final official estimates, uncertainty measures for the official estimates are not available for publication. Erciulescu et al. (2016) proposed a model-based estimation approach that incorporates multiple sources of information to produce county-level and ASD-level acreage estimates and associated measures of uncertainty. The authors investigated different benchmarking methods for acreage totals and Nandram et al. (2016) proposed a robust Bayesian benchmarking method for model-based estimates. Although the model-based methods in Erciulescu et al. (2016) and Nandram et al. (2016) could also be applied to yield, it is important to evaluate the effects of the different benchmarking adjustments for yield, as the ratio of production and acreage.

To introduce the problem setting and notation, let $\theta_{ij}^{T1}, \theta_{ij}^{T2}$ and θ_{ij}^{R} denote the true (unknown) parameters of interest for the j^{th} subarea, in the i^{th} area (a group of subareas), corresponding to two totals and their ratio, respectively. Let a_{T1}, a_{T2} and $a_R := \frac{a_{T1}}{a_{T2}}$ denote the fixed targets for the totals and for the ratio, respectively. Then, the benchmarking constraints to be satisfied are

$$\begin{aligned} \sum_{j=1}^{n_{c_i}} w_{ij}^{\zeta} \hat{\theta}_{ij}^{\zeta,B} &= \hat{\theta}_i^{\zeta,B}, \\ \sum_{i=1}^{m} w_i^{\zeta} \hat{\theta}_i^{\zeta,B} &= a_{\zeta}, \end{aligned} \tag{1}$$

where $\hat{\theta}_{ij}^{\zeta,B}$ and $\hat{\theta}_i^{\zeta,B}$ denote the final estimates for the subareas and for the areas, respectively, for $\zeta \in \{T1, T2, R\}$, $j = 1, ..., n_{c_i}$ and $i = 1, ..., m$.

Without loss of generality, the methodology is illustrated for the subarea-to-target constraints, using subscript j to denote the subarea,

$$\sum_{j=1}^{n_c} w_j^{\zeta} \hat{\theta}_j^{\zeta,B} = a_{\zeta}, \tag{2}$$

where $n_c = \sum_{i=1}^{m} n_{c_i}$ is the total number of subareas. Area-level estimates are constructed using the information for all $j = 1, ..., n_{c_i}$ in area i. Different scenarios of estimating and benchmarking the triplet $(\theta_j^{T1}, \theta_j^{T2}, \theta_j^{R})$, using MCMC, are considered. It is shown that under some scenarios, the estimates may fail the benchmarking constraints at all the aggregation levels, while under other scenarios, the estimates may lead to an inconsistent ratio definition, $\hat{\theta}_j^{R,B} = \frac{\hat{\theta}_j^{T1,B}}{\hat{\theta}_j^{T2,B}}$, for all $j = 1, ..., n_c$. Variance estimation is discussed for each scenario.

Model-based ratio estimates may be constructed using survey ratio estimates and auxiliary data. For this approach, an external benchmarking constraint is that the weighted sum of the estimates equals a constant, as defined in (2), and the benchmarking weight, w_j^R, applied to the ratio estimate is a function of the denominator estimate, $\hat{\theta}_j^{T2,B}$.

Two cases are considered. The first case is inspired by Nandram et al. (2014) and Berg et al. (2014), where model-based estimates for a ratio (in their applications, yield or cash rental rate) are benchmarked conditioning on the denominator estimates. Alternatively, we introduce the second case, when the benchmarking weights for the ratios are functions of the MCMC iterates for the denominator estimates. In the first case, the authors construct estimates for the ratio only, not for the triplet, and the benchmarking weights for the ratios are functions of the denominator estimates, treated as fixed and known, and estimates for the numerator are constructed as a result. The numerator estimates are the product of the ratio estimates and the denominator estimates, $\hat{\theta}_j^{T1,B} = \hat{\theta}_j^{R,B} \hat{\theta}_j^{T2,B}$, with estimated variances conditional on the denominator estimates. In the second case, the numerator estimates are constructed using the product of the MCMC iterates for the ratio and for the denominator. The variance estimation for the numerator estimates is improved since conditioning on the denominator estimates is not necessary.

This paper is structured in two main sections. In Section 2, we introduce the methods of applying the benchmarking constraints to the triplet (numerator, denominator, ratio) estimates and discuss the advantages and disadvantages for each method. Estimates for subarea-level and for area-level are constructed for the two cases described above. In Section 3, we illustrate the methods described in Section 2, to produce end-of-season, model-based, county-level and ASD-level acreage, production and yield estimates, for 2014 corn and soybeans, in selected states. A discussion is provided in Section 4.

2. Benchmarking a Ratio using Markov chain Monte Carlo Methods

We model the denominator total $T2$ and the ratio R independently, and derive the numerator total $T1$ as a result. Estimating two quantities and deriving the third is a method consistent with the data collection, as a survey respondent may provide acreage and production or yield on the survey questionnaire. Although the methods apply to any benchmarking adjustments, in the following sections, we provide illustrations for the ratio benchmarking (see Rao and Molina 2015, Section 6.4.6).

Assume that the Bayes estimates, which satisfy the benchmarking constraints in (2), are constructed using MCMC methods and denoted by $\hat{\theta}_j^{T1,B}$, $\hat{\theta}_j^{T2,B}$ and $\hat{\theta}_j^{R,B}$, for subarea $j = 1, ..., n_c$. Since the benchmarking weights for the ratio depend on the denominator, Bayes estimates for the denominator need to be constructed first; the numerator will be derived.

Two independent Bayesian models are fit to the denominator total and the ratio survey estimates, respectively. Results are constructed using the chains of MCMC iterates for the two quantities: $T2$ and R. Initial fitting of the models results in iterates that do not satisfy (2): let $\theta_{jk}^{T2,noadj}$ and $\theta_{jk}^{R,noadj}$ denote the iterates of θ_j^{T2} and θ_j^{R}, respectively, where $k = 1, ..., K$. For each of the two parameters $\zeta \in \{T2, R\}$, at every subarea $j = 1, ..., n_c$, an MCMC sampler is executed to get:

$$
\begin{array}{c}
\begin{array}{cc}
\begin{array}{c} j\backslash k \\ 1 \\ 2 \\ \vdots \\ n_c \end{array}
&
\begin{array}{cccc} 1 & 2 & & K \\ \end{array}
\end{array}
\\
\left(
\begin{array}{cccc}
\theta_{11}^{\zeta,noadj} & \theta_{12}^{\zeta,noadj} & \dots & \theta_{1K}^{\zeta,noadj} \\
\theta_{21}^{\zeta,noadj} & \theta_{22}^{\zeta,noadj} & \dots & \theta_{2K}^{\zeta,noadj} \\
\vdots & \vdots & \vdots & \vdots \\
\theta_{n_c1}^{\zeta,noadj} & \theta_{n_c2}^{T1/T2/R,noadj} & \dots & \theta_{n_cK}^{\zeta,noadj}
\end{array}
\right)
\mapsto \hat{\theta}_j^{\zeta,noadj} = K^{-1} \sum_k \theta_{jk}^{\zeta,noadj}, \text{ for fixed } j,
\\
\downarrow
\\
\sum_j w_{jk}^{\zeta} \theta_{jk}^{\zeta,noadj} \stackrel{?}{=} a_\zeta, \text{ for fixed } k,
\end{array}
$$

where (j, k) denotes the k^{th} iterate for subarea j, $k = 1, ..., K$. Hereafter, it is assumed that the benchmarking weights for the denominator total, w_j^{T2}, are fixed and known (do not depend on k). So $w_{jk}^{T2} = w_j^{T2}$, for all $k = 1, ..., K$.

Note that the duplet $(\hat{\theta}_j^{T2,noadj}, \hat{\theta}_j^{R,noadj}), j = 1, ..., n_c$, is a set of estimators for the denominator and ratio, but the benchmarking constraints in (2) may not hold. Also, the estimator for the numerator remains to be constructed. Moreover, the benchmarking constraints in (2) may not hold at the iteration level. Choices for the weights w_{jk}^{R} that ensure the constraints in (2) and the identity between the ratio and the totals, at the iteration level, after benchmarking adjustment, remain to be constructed.

Bayes estimators $\hat{\theta}_j^{T2,B}$ and $\hat{\theta}_j^{R,B}$ are constructed using the MCMC iterates from fitting two models, for the denominator and for the ratio, independently. Final estimates for the numerator total $\hat{\theta}_j^{T1,B}$ are derived as a result. The Monte Carlo estimates of the denominator and the ratio, $E(\theta_j^{T2}|data)$ and $E(\theta_j^{R}|data)$, are constructed using the traditional arithmetic mean of the iterates:

$$
\begin{aligned}
K^{-1} \textstyle\sum_k^K \theta_{jk}^{T2,noadj} &= \hat{\theta}_j^{T2,noadj}, \\
K^{-1} \textstyle\sum_k^K \theta_{jk}^{R,noadj} &= \hat{\theta}_j^{R,noadj},
\end{aligned} \tag{3}
$$

where $\hat{\theta}_j^{T2,noadj}$ and $\hat{\theta}_j^{R,noadj}$ are the Monte Carlo estimators of θ_j^{T2} and θ_j^{R}, respectively, before the benchmarking adjustment.

The benchmarking constraints given in (2) are used to adjust the Bayes estimates, computed at the k^{th} iteration of the MCMC. For example, for a fixed k and a *ratio* benchmarking adjustment,

$$\begin{aligned} \theta_{jk}^{T2} &= \theta_{jk}^{T2,noadj} \frac{a_{T2}}{\sum_{j=1}^{n_c} w_{jk}^{T2} \theta_{jk}^{T2,noadj}}, \\ \theta_{jk}^{R} &= \theta_{jk}^{R,noadj} \frac{a_{R}}{\sum_{j=1}^{n_c} w_{jk}^{R} \theta_{jk}^{R,noadj}}. \end{aligned} \tag{4}$$

Note that for each iteration k, the subarea-level iterates aggregate to the target values,

$$\begin{aligned} \textstyle\sum_{j=1}^{n_c} w_{jk}^{T2} \theta_{jk}^{T2} &= \textstyle\sum_{j=1}^{n_c} w_{jk}^{T2} \theta_{jk}^{T2,noadj} \frac{a_{T2}}{\sum_{j=1}^{n_c} w_{jk}^{T2} \theta_{jk}^{T2,noadj}} = a_{T2}, \\ \textstyle\sum_{j=1}^{n_c} w_{jk}^{R} \theta_{jk}^{R} &= \textstyle\sum_{j=1}^{n_c} w_{jk}^{R} \theta_{jk}^{R,noadj} \frac{a_{R}}{\sum_{j=1}^{n_c} w_{jk}^{R} \theta_{jk}^{R,noadj}} = a_{R}. \end{aligned} \tag{5}$$

If θ_{jk}^{T2} and θ_{jk}^{R} are the adjusted iterates, using *any* benchmarking method, then the posterior subarea means are estimated by

$$\begin{aligned} \widehat{E}(\theta_j^{T2}|\text{data and constraint})] &= K^{-1} \textstyle\sum_k^K \theta_{jk}^{T2} := \hat{\theta}_j^{T2,B}, \\ \widehat{E}(\theta_j^{R}|\text{data and constraint}) &= K^{-1} \textstyle\sum_k^K \theta_{jk}^{R} := \hat{\theta}_j^{R,B}, \end{aligned} \tag{6}$$

and the posterior subarea variances are estimated by

$$\begin{aligned} \widehat{Var}(\theta_j^{T2}|\text{data and constraint}) &= (K-1)^{-1} \textstyle\sum_k^K \left(\theta_{jk}^{T2} - K^{-1} \sum_k^K \theta_{jk}^{T2}\right)^2, \\ \widehat{Var}(\theta_j^{R}|\text{data and constraint}) &= (K-1)^{-1} \textstyle\sum_k^K \left(\theta_{jk}^{R} - K^{-1} \sum_k^K \theta_{jk}^{R}\right)^2. \end{aligned} \tag{7}$$

Using the adjusted subarea-level iterates θ_{jk}^{T2} and θ_{jk}^{R} (for example, defined in (4)), for all $j = 1, ..., n_{c_i}$ within area i, define the adjusted area-level iterates

$$\begin{aligned} \theta_{ik}^{T2} &= \textstyle\sum_{j \in d_i} w_{jk}^{T2} \theta_{jk}^{T2}, \\ \theta_{ik}^{R} &= \textstyle\sum_{j \in d_i} w_{jk}^{R} \theta_{jk}^{R}, \end{aligned} \tag{8}$$

where d_i is the set of all subareas within area i. Then the posterior area means are estimated by

$$\begin{aligned} \widehat{E}(\theta_i^{T2}|\text{data and constraint}) &= K^{-1} \textstyle\sum_k^K \theta_{ik}^{T2} := \hat{\theta}_i^{T2,B}, \\ \widehat{E}(\theta_i^{R}|\text{data and constraint}) &= K^{-1} \textstyle\sum_k^K \theta_{ik}^{R} := \hat{\theta}_i^{R,B}, \end{aligned} \tag{9}$$

and the posterior area variances are estimated by

$$\begin{aligned} \widehat{Var}(\theta_i^{T2}|\text{data and constraint}) &= (K-1)^{-1} \textstyle\sum_k^K \left(\theta_{ik}^{T2} - K^{-1} \sum_k^K \theta_{ik}^{T2}\right)^2, \\ \widehat{Var}(\theta_i^{R}|\text{data and constraint}) &= (K-1)^{-1} \textstyle\sum_k^K \left(\theta_{ik}^{R} - K^{-1} \sum_k^K \theta_{ik}^{R}\right)^2. \end{aligned} \tag{10}$$

Since the benchmarking weights for the ratio are functions of the denominator total, we consider two cases for the benchmarking adjustment at the (MCMC) iteration level. In the first case, $\hat{\theta}_j^{T2,B}$ are treated as fixed and known, leading to fixed weights $w_{jk}^{R} = w_j^{R}$, for all $k = 1, ..., K$. Two examples of benchmarking using fixed weights are Berg et al. (2014) and Nandram et al. (2014). Alternatively, w_{jk}^{R} are constructed using the iterates of θ_j^{T2} and $w_j^{R} = K^{-1} \sum_k w_{jk}^{R}$. Both cases ensure the benchmarking constraints at the iteration level, but only the first case at the subarea level. Similar choices of weights are investigated for the area-level estimates.

2.1 Fixed benchmarking weights (R11)

For this case, the denominator total subarea-level estimates $\hat{\theta}_j^{T2,B}$ and area-level estimates $\hat{\theta}_i^{T2,B}$ are constructed as in (6, 9), and treated as fixed and known. Then, the subarea-level bechmarking weights w_{jk}^R are set equal to the ratio of the subarea-level denominator total to the target denominator total $w_j^R := \frac{\hat{\theta}_j^{T2,B}}{a_{T2}}$, for all $k = 1, ..., K$. The bechmarking weights w_{jk}^R are applied to the iterates $\theta_{jk}^{R,noadj}$, for example as in (4), leading to the adjusted subarea-level ratio estimator $\hat{\theta}_j^{R,B}$. Similarly, the area-level benchmarking weights w_{ik}^R are set equal to $w_i^R := \frac{\hat{\theta}_i^{T2,B}}{a_{T2}}$, for all $k = 1, ..., K$, and applied to the iterates $\theta_{ik}^{R,noadj}$, for example as in (8), leading to the adjusted area-level ratio estimator $\hat{\theta}_i^{R,B}$.

Subarea-level iterates for the numerator are constructed as

$$\theta_{j,k}^{T1} := \hat{\theta}_j^{T2,B} \theta_{j,k}^R, \tag{11}$$

and the estimator $\hat{\theta}_j^{T1,B}$ is constructed as

$$\widehat{E}(\theta_j^{T1} | \hat{\theta}_j^{T2,B}, \text{data and constraint}) = K^{-1} \sum_k \hat{\theta}_j^{T2,B} \theta_{j,k}^R = \hat{\theta}_j^{T2,B} \hat{\theta}_j^{R,B} := \hat{\theta}_j^{T1,B},$$

automatically satisfying the benchmarking constraint (see Appendix A1), with

$$\begin{aligned} \widehat{Var}(\theta_j^{T1} | \hat{\theta}_j^{T2,B}, \text{data and constraint}) &= (K-1)^{-1} \textstyle\sum_k \left(\hat{\theta}_j^{T2,B} \theta_{j,k}^R - K^{-1} \sum_k \hat{\theta}_j^{T2,B} \theta_{j,k}^R \right)^2 \\ &= (K-1)^{-1} \left(\hat{\theta}_j^{T2,B} \right)^2 \textstyle\sum_k \left(\theta_{j,k}^R - K^{-1} \sum_k \theta_{j,k}^R \right)^2 \\ &= \left(\hat{\theta}_j^{T2,B} \right)^2 \widehat{Var}(\theta_j^R | \text{data and constraint}). \end{aligned}$$

Area-level iterates for the numerator are constructed as

$$\theta_{i,k}^{T1} := \hat{\theta}_i^{T2,B} \theta_{i,k}^R, \tag{12}$$

and the estimator $\hat{\theta}_i^{T1,B}$ is constructed as

$$\widehat{E}(\theta_i^{T1} | \hat{\theta}_i^{T2,B}, \text{data and constraint}) = K^{-1} \sum_k \hat{\theta}_i^{T2,B} \theta_{i,k}^R = \hat{\theta}_i^{T2,B} \hat{\theta}_i^{R,B} := \hat{\theta}_i^{T1,B},$$

satisfying automatically the benchmarking constraint (see Appendix A1), with

$$\begin{aligned} \widehat{Var}(\theta_i^{T1} | \hat{\theta}_i^{T2,B}, \text{data and constraint}) &= (K-1)^{-1} \textstyle\sum_k \left(\hat{\theta}_i^{T2,B} \theta_{i,k}^R - K^{-1} \sum_k \hat{\theta}_i^{T2,B} \theta_{i,k}^R \right)^2 \\ &= (K-1)^{-1} \left(\hat{\theta}_i^{T2,B} \right)^2 \textstyle\sum_k \left(\theta_{i,k}^R - K^{-1} \sum_k \theta_{i,k}^R \right)^2 \\ &= \left(\hat{\theta}_i^{T2,B} \right)^2 \widehat{Var}(\theta_i^R | \text{data and constraint}). \end{aligned}$$

Note that, while the benchmarking constraints are satisfied for all three parameters of interest, the MCMC summaries for the numerator are conditional on the denominator estimates; see (11).

Remark. Since the estimates for the denominator are treated as fixed and known, the method of producing these estimates has no effect on the final ratio estimates. However, for the benchmarking constraints in (2) to hold, the set of estimates for the denominator needs to satisfy the corresponding benchmarking constraint, too. For example, survey direct estimates for the denominator may not satisfy the external benchmarking constraint in (2), and using them to construct benchmarking weights for the ratio will lead to inconsistencies.

2.2 Random benchmarking weights (R12)

For this case, subarea-level benchmarking weights w_{jk}^{R} are set equal to the ratio of the weighted denominator total iterates to the denominator total target $\frac{w_{jk}^{T2}\theta_{jk}^{T2}}{a_{T2}} = \frac{w_{j}^{T2}\theta_{jk}^{T2}}{a_{T2}}$. The benchmarking weights w_{jk}^{R} are applied to the iterates $\theta_{jk}^{R,noadj}$, leading to the benchmarked iterates θ_{jk}^{R} and to the adjusted ratio estimator $\hat{\theta}_{j}^{R,B}$. Iterates for the numerator total θ_{jk}^{T1} are constructed as the product of the iterates for the denominator total θ_{jk}^{T2} and the iterated for the ratio θ_{jk}^{R},

$$\theta_{jk}^{T1} := \theta_{jk}^{T2}\theta_{jk}^{R}, \tag{13}$$

and the estimator $\hat{\theta}_{j}^{T1,B}$ is constructed as

$$\widehat{E}(\theta_{j}^{T1}|\text{data and constraint}) = K^{-1}\sum_{k}^{K}\theta_{jk}^{T1} := \hat{\theta}_{j}^{T1,B},$$

satisfying automatically the benchmarking constraint (see Appendix A2), with

$$\begin{aligned}\widehat{Var}(\theta_{j}^{T1}|\text{data and constraint}) &= (K-1)^{-1}\textstyle\sum_k\left(\theta_{jk}^{T1} - K^{-1}\sum_k\theta_{jk}^{T1}\right)^2\\ &= (K-1)^{-1}\textstyle\sum_k\left(\theta_{jk}^{T2}\theta_{jk}^{R} - K^{-1}\sum_k\theta_{jk}^{T2}\theta_{jk}^{R}\right)^2.\end{aligned}$$

Under the choice of random weights, the benchmarking constraints are satisfied, at the iteration level, for the three parameters of interest and the MCMC summaries for the numerator are not conditional on the denominator estimates; see (13). However, the benchmarking constraints for the ratio are not (exactly) satisfied at the subarea and area levels, and the equality between the subarea-level ratio estimator and the ratio of the subarea-level estimators for the totals is not (exactly) satisfied; the differences in the subarea-level estimates are illustrated in Appendix A2.

The area-level estimators for the numerator are derived in a similar way the subarea-level estimators were derived. The area-level weights w_{ik}^{R} are set equal to $\frac{\theta_{ik}^{T2}}{a_{T2}}$ and applied to the iterates $\theta_{ik}^{R,noadj}$, leading to the benchmarked iterates θ_{ik}^{R} and to the adjusted ratio estimator $\hat{\theta}_{i}^{R,B}$. Iterates θ_{ik}^{T1} are constructed as the product of θ_{ik}^{T2} and θ_{ik}^{R},

$$\theta_{ik}^{T1} := \theta_{ik}^{T2}\theta_{ik}^{R}, \tag{14}$$

and the estimator $\hat{\theta}_{i}^{T1,B}$ is constructed as

$$\widehat{E}(\theta_{i}^{T1}|\text{data}) = K^{-1}\sum_{k}^{K}\theta_{ik}^{T1} := \hat{\theta}_{i}^{T1,B},$$

satisfying automatically the benchmarking constraint (see Appendix A2), with

$$\begin{aligned}\widehat{Var}(\theta_{i}^{T1}|\text{data and constraint}) &= (K-1)^{-1}\textstyle\sum_k\left(\theta_{ik}^{T1} - K^{-1}\sum_k\theta_{ik}^{T1}\right)^2\\ &= (K-1)^{-1}\textstyle\sum_k\left(\theta_{ik}^{T2}\theta_{ik}^{R} - K^{-1}\sum_k\theta_{ik}^{T2}\theta_{ik}^{R}\right)^2.\end{aligned}$$

3. Application to the End-of-Season Crop Estimates at NASS

Model-based estimates for 2014 end-of-season crop acreage, production and yield are constructed for corn and soybeans in selected states. Survey direct estimates are available from the NASS's quarterly crops Acreage, Production and Stocks (APS) surveys and its supplement, the County Agricultural Production Survey (CAPS). The goal is to produce county-level and ASD-level estimates for three parameters of interest: harvested acreage total, production total and yield, where yield is defined as the ratio of production to harvested acreage, such that the benchmarking constraints between the three levels (county, agricultural statistics district and state) hold. The state-level values are prepublished and considered as fixed targets in the benchmarking. Estimation is conducted state by state and commodity by commodity.

3.1 Direct Survey Estimates

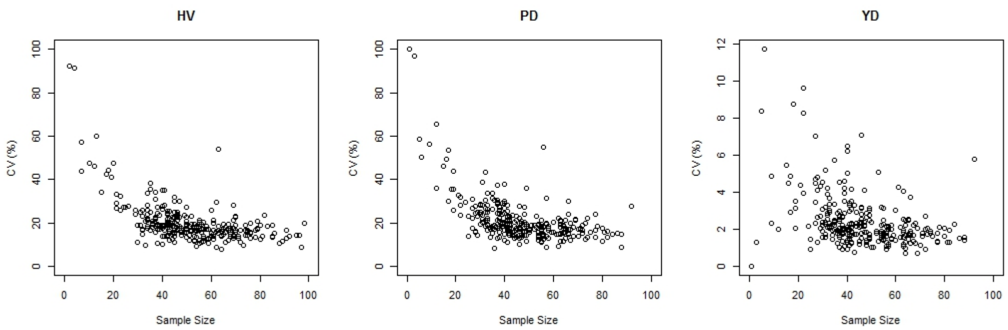
Three states that are major producers of corn and soybeans are considered in this application: Illinois, Indiana and Iowa. The number of counties in each state equals 102, 92 and 99, respectively. The number of ASDs is 9 in each of the three states. Official state-level values of acreage, production and yield are obtained from NASS QuickStats, at USDA NASS (2017). The CAPS summary provides the county-level direct survey estimates and their corresponding sampling variances, for the three parameters of interest, the ratio definition being satisfied for the three sets of survey estimates; i.e., yield survey estimates equal to the ratio of production survey estimates to harvested acreage survey estimates. However the benchmarking constraints in (2) do not hold for the survey estimates. The number of available survey estimates for each parameter differs across years, states and commodities. For example, fewer county-level survey estimates are available for yield than for harvested acreage. This is not the case for the 2014 corn and soybeans estimates in Illinois, Indiana and Iowa, as survey estimates are available for the three parameters, for all the counties.

The county sample size is the number of records used to construct the county-level survey estimates; it differs from state to state, commodity to commodity, and parameter to parameter, taking values in the range 1 to 98 for corn in Illinois, Indiana and Iowa. The production and yield sample sizes are equal and are less than or equal to the acreage sample size. The estimated CVs for the survey estimates increase as the county sample size decreases, and their ranges also differ from state to state, commodity to commodity, and parameter to parameter. For harvested acreage (HV) survey estimates, the CVs range from 8.1% to 92.3%, for production (PD) survey estimates, the CVs range from 8.6% to 100.0% and for yield (YD) survey estimates, the CVs range from 0.0% to 11.7%. Figure 1 shows the distribution of the coefficients of variation (CVs) of the county-level corn survey estimates, relative to the county-level corn sample size. Similar CVs and sample sizes are identified for soybeans.

3.2 Auxiliary Sources of Information

We select three sources of auxiliary information for covariates: the USDA Farm Service Agency's planted acreage administrative data (FSA.PL), the USDA Natural Resources Conservation Service's National Commodity Crop Productivity Index (NCCPI) and the National Oceanic and Atmospheric Administration's weather data. In particular, from the set of weather variables, we select the March Precipitation (NOAA March PCPN). The county-level covariates, FSA.PL and NCCPI, were selected from a larger pool of variables, using

Figure 1: **County**-level Direct Survey Estimates, CVs versus Sample Sizes
2014 **Corn** in Illinois, Indiana and Iowa



The plots illustrate the relationship between the coefficients of variation (CVs) of the county-level corn survey estimates and the county-level corn sample size, for harvested acreage (HV), production (PD) and yield (YD). The CV ranges and the sample size ranges ***differ*** *from state to state, from parameter to parameter and from commodity to commodity.*

correlation analysis and model comparison criteria. As expected, exploratory data analysis shows that strong linear relationships exist between the survey acreage and production estimates, and the acreage administrative data, and that strong linear relationships exist between the survey yield estimates and the NCCPI values. The ASD-level covariate, NOAA March PCPN, was also selected based on correlation analysis and model comparison criteria. It is also known that March precipitation may impact the corn planting practices (see Erciulescu et al. (2016)) and March is the first month by which corn and soybeans planting starts, or is completed, across the United States, see USDA NASS (2010).

3.3 A Hierarchical Bayesian Model

Following Erciulescu et al. (2016), the proposed model is a subarea-level model, where the area represents the ASD and the subarea represents the county. Of interest is estimation of the triplet (production, acreage and yield) at the county and ASD levels, such that the county-level estimates, the ASD-level estimates and the state-level values agree.

Let $i = 1, ..., m$ be an index for the m ASDs in the state, $j = 1, ..., n_{c_i}$, be an index for the n_{c_i} counties in ASD i, and n_{ij} be the county sample size of the j^{th} county in the i^{th} ASD. The total number of counties in the state is $\sum_{i=1}^{m} n_{c_i} = n_c$ and the state sample size is $\sum_{i=1}^{m} \sum_{j=1}^{n_{c_i}} n_{ij} = n$. The county-level covariate values are x_{ij} and the ASD-level covariate values are z_i.

Let $\tilde{\theta}_{ij}$ be the survey estimate in county i and ASD j, σ_{ij}^2 be the sampling variance in county i and ASD j, and c_{ij} be known constants (to be specified). Illustrated for one state, one commodity and one parameter, let the hierarchical Bayesian subarea-level model be

$$\begin{aligned} c_{ij}^{-1}\tilde{\theta}_{ij}|(\boldsymbol{\beta}, u_{ij}, v_i, \sigma_{ij}^2) &\overset{ind}{\sim} N((1, x_{ij}, z_i)\boldsymbol{\beta} + u_{ij} + v_i, c_{ij}^{-2}\sigma_{ij}^2), \\ \theta_{ij} &:= (1, x_{ij}, z_i)\boldsymbol{\beta} + u_{ij} + v_i, \\ u_{ij}|\sigma_u^2 &\overset{iid}{\sim} N(0, \sigma_u^2), \\ v_i|\sigma_v^2 &\overset{iid}{\sim} N(0, \sigma_v^2). \end{aligned} \tag{15}$$

Model (15) borrows information from all the counties in an ASD and from all the ASDs in the state, while combining auxiliary information available at different levels of aggregation. Note that model (15) with $c_{ij} = 1$, known σ_{ij}^2 and $\sigma_v^2 = 0$, reduces to the area-level model, introduced by Fay and Herriot (1979).

It is important to note that the distribution of the county-level total acreage estimates is skewed. To reduce skewness and improve the symmetry of the distribution of the survey estimates, the model fit to acreage uses $c_{ij} = n_{ij}$. On the other hand, the distribution of the county-level yield estimates is approximately symmetric, and the survey estimates are modeled without scaling. That is, when the model is fit to yield, $c_{ij} = 1$.

In model (15), the sampling variances σ_{ij}^2 are fixed. Often, estimates $\tilde{\sigma}_{ij}^2$ are available, or can be computed, from the survey. In order to reduce the variability of the sampling variances and the relationship with the survey estimates, we consider a model for the variance of the acreage estimates that is not applicable to yield.

Let the model for the sampling variances be

$$\begin{aligned} (n_{ij} - 1)\frac{\tilde{\sigma}_{ij}^2}{\sigma_{ij}^2}|\sigma_{ij}^2 &\overset{ind}{\sim} \chi^2_{(n_{ij}-1)}, \\ log(c_{ij}^{-2}\sigma_{ij}^2)|(\boldsymbol{\alpha}, \sigma^2) &\overset{ind}{\sim} N((1, log(x_{ij}))\boldsymbol{\alpha}, \sigma^2). \end{aligned} \tag{16}$$

Wang and Fuller (2003), You and Chapman (2006), Gonzalez-Manteiga, et al. (2010), Erciulescu and Berg (2014) consider the case where the sampling variances are unknown and modeled separately. Note that the logarithmic transformation can be applied to the auxiliary acreage estimates because they are strictly positive.

To complete the Bayesian model specification, we consider a priori independent parameters and adopt noninformative, proper priors for $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \sigma_u^2, \sigma_v^2, \sigma^2)$. The prior distributions for the model parameters $(\boldsymbol{\beta}', \boldsymbol{\alpha}')$ are normal distributions with mean and variance denoted by the least squares estimates of $(\boldsymbol{\beta}', \boldsymbol{\alpha}')$. The least squares estimates of $\boldsymbol{\beta}$ are obtained from fitting a simple linear model for the county-level survey estimates against the county-level auxiliary information. The least squares estimates of $\boldsymbol{\alpha}$ are obtained from fitting a simple linear model for the county-level sampling variances against the county-level auxiliary information, on the logarithmic scale. The prior distributions for the model variance components σ_u^2, σ_v^2, and σ^2 are $Uniform(0, 10^8), Uniform(0, 10^8)$, and $Inverse-Gamma(10^3, 10^3)$, respectively. We acknowledge that the $Inverse-Gamma$ distribution is generally a natural prior choice for a variance parameter because of its conjugacy. However, in a hierarchical model, the $Uniform$ distribution performs better as a prior for the random-effects variance component, see Browne and Draper (2005).

3.3.1 Model Fit and Estimation

For each state and for each commodity, model (15, 16) is fit, independently, to the CAPS survey estimates of harvested acreage per unit, and model (15) is fit to the CAPS survey estimates of yield. The harvested acreage total, the production total and the yield ratio correspond to the $T2, T1$ totals and to the R ratio in Section 2, respectively. As a result of variable selection, each model is fit using one county-level covariate and one ASD-level covariate. The county-level covariate for acreage is the administrative acreage, scaled by the county-level sample size. The county-level covariate for yield is the NCCPI. The ASD-level covariate is the set of March precipitation values available from NOAA.

The models are fit using R JAGS, and the posterior distributions constructed using MCMC simulation. We use $10,000$ Monte Carlo samples and $1,000$ burn-in samples, 3 chains, each thinned every 9 samples. The convergence is monitored using trace plots, the multiple potential scale reduction factors (values less than 1.1) and the Geweke test of stationarity for each chain, see Gelman and Rubin (1992) and Geweke (1992). Also, once the simulated chains have mixed, we construct the effective number of independent simulation draws to monitor simulation accuracy.

Using the chains of iterates obtained from the model fit, we construct posterior summaries from the posterior distributions of the county-level and ASD-level estimates, under the different benchmarking scenarios presented in Section (2), treating the ij index as the j index. The classic random benchmarking adjustment (see Rao and Molina 2015) is applied to county-level estimates, under the constraint to the fixed, prepublished state-level values. The benchmarking weights for yield are functions of the harvested acreage estimates. Let θ_j^R denote the yield parameter at the county-level, let θ_j^{T2} denote the harvested acreage parameter at the county-level and let θ_j^{T1} denote the production parameter at the county-level. Then the benchmarking weights for the acreage w_{jk}^{T2} are set equal to the county-level sample sizes, for all $k = 1, ..., K$, because the models are fit to the scaled totals. Hence, the benchmarking weights for production equal to the benchmarking weights for acreage $w_j^{T1} = w_j^{T2} = w_{jk}^{T2}$, for all $k = 1, ..., K$. The model-based point estimates are compared to the corresponding NASS official estimates (ASB), and the model-based standard errors of the point estimates are compared to the NASS survey errors.

3.4 Results

We will present results for the model-based estimates for harvested acreage (HV), the model-based estimates for yield (YD), and the derived estimates for production (PD), as described in Section 2. The MCMC iterates for HV are used to construct benchmarking weights for yield (YD) estimates.

The median relative differences between the HV model-based estimates, ratio adjusted to satisfy benchmarking constraints, and the ASB estimates, are 3.54% for corn and 4.46% for soybeans, which are approximately equal to 0.43 and 0.54 standard errors for corn and soybeans, respectively. Moreover the estimated error in the model-based HV estimates is lower than the estimated error in the survey HV estimates; the model-based estimates have variances approximately 50% lower than the variances of the survey estimates for both corn and soybeans. As a result, the estimated CVs for the derived county-level production (R12) estimates are approximately $40 - 45\%$ of the corresponding estimated CVs for the county-level survey estimates. This is a result of the model's strength of borrowing information across counties and ASDs, and from auxiliary data.

We compare the production (PD) and yield (YD) model-based estimates, under the different ratio benchmarking applications presented in Section 2, to the survey estimates and to the published estimates. To protect the confidentiality of the data, we present three metrics to quantify the relative differences between the model-based estimates (MERB) and the ASB estimates, and the relative differences between the standard errors of the model-based estimates (MERBSE) and of the survey estimates (CAPSSE). Results for the 2014 county-level corn estimates in Illinois, Indiana and Iowa are presented in Table 1 and Figure 2.

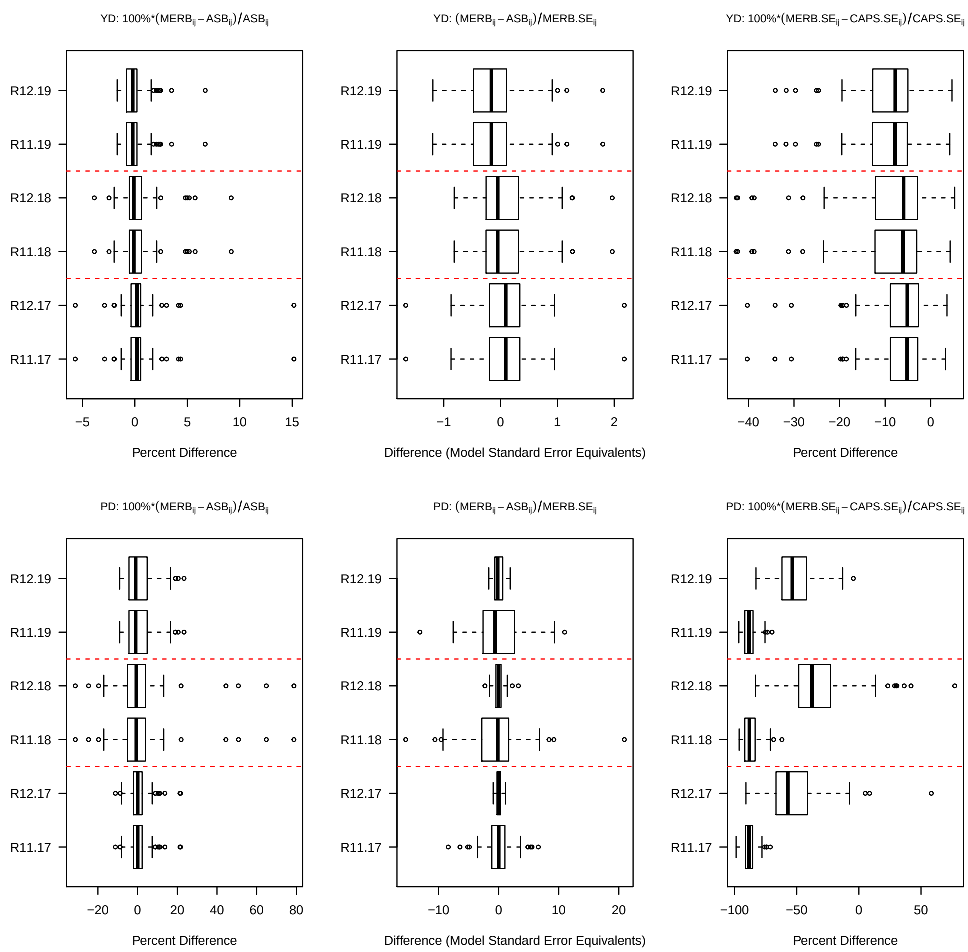
Table 1: Median Absolute Relative Differences of **County**-Level Estimates
2014 **Corn and Soybeans** in Illinois, Indiana and Iowa

			$median_{ij} \left\lvert \frac{MERB_{ij} - ASB_{ij}}{ASB_{ij}} \right\rvert$		$median_{ij} \left\lvert \frac{MERB_{ij} - ASB_{ij}}{MERBSE_{ij}} \right\rvert$		$median_{ij} \left\lvert \frac{MERBSE_{ij} - CAPSSE_{ij}}{CAPSSE_{ij}} \right\rvert$	
	State	Estimator	PD	YD	PD	YD	PD	YD
Corn	IL	R11	2.17%	0.50%	1.14	0.30	88.52%	5.37%
		R12	2.17%	0.50%	0.31	0.29	58.13%	5.32%
	IN	R11	4.49%	0.59%	2.12	0.27	88.11%	6.08%
		R12	4.48%	0.59%	0.43	0.27	38.83%	5.95%
	IA	R11	4.58%	0.56%	2.61	0.34	88.39%	7.84%
		R12	4.58%	0.56%	0.65	0.34	53.59%	7.78%
Soybeans	IL	R11	3.12%	0.55%	1.37	0.26	86.47%	3.69%
		R12	3.13%	0.55%	0.37	0.26	49.63%	3.55%
	IN	R11	6.76%	0.82%	3.05	0.44	86.84%	3.47%
		R12	6.76%	0.82%	0.61	0.44	38.31%	3.64%
	IA	R11	5.91%	0.97%	3.21	0.52	87.34%	6.16%
		R12	5.91%	0.97%	0.83	0.52	54.22%	6.20%

The YD and PD point estimates are similar when benchmarked using fixed and random weights because the variance of the random weights is negligible (in the range $(10^{-6}, 10^{-7})$). Although, under random weights, the equality between the yield estimates and the ratio of production estimates to harvested acreage estimates is not exactly satisfied, the percentage medians of relative differences, relative to the ASB estimates are equal for the two cases (R11 and R12), up to two significant digits. Most of the YD county-level estimates are within $0 - 1\%$ from the corresponding ASB estimates, or within $0 - 0.52$ standard errors from the corresponding ASB estimates; 99.7% of the ASB estimates fall inside the 95% credible intervals of the corresponding model-based estimates. Also, most of the PD county-level estimates are within $3 - 7\%$ from the corresponding ASB estimates, or within $0 - 3.21$ standard errors from the corresponding ASB estimates; 52% of the ASB estimates fall inside the 95% credible intervals of the corresponding model-based estimates. However, the variance estimates for PD (R11) are artificially decreased, as a result of conditioning on the acreage estimates. Hence, there is a significant difference between the model-based PD estimates and the ASB estimates.

Results for the 2014 ASD-level corn estimates in Illinois, Indiana and Iowa, are presented in Table 2 and Figure 3. The benchmarking applications presented in Section 2 perfom similar for the ASD-level estimates and for the county-level estimates. The absolute relative differences between the model-based ASD-level estimates and the corresponding official estimates are smaller than the absolute relative differences for the county-level; this is expected because the survey direct estimates are also more reliable at the ASD-level than at the county-level due to larger sample sizes at the ASD-level than at the county-level.

Figure 2: **County**-level Model-based Estimates versus ASB Estimates
2014 **Corn** in Illinois, Indiana and Iowa



The box plots illustrate the relationship between the county-level direct survey estimates, the model-based estimates (R11, R12) and the ASB estimates, for production and yield. The model-based estimates are ratio benchmarked using the methods described in Section 2. The three horizontal sets of box plots correspond to the three states, denoted by their FIPS codes (17 for Illinois, 18 for Indiana and 19 for Iowa). One county with sample size equal to one was removed from the visual representations because its (approximately zero) corresponding survey variance decreased the visual precision. Results are similar for soybeans.

3.5 Other results

Following Berg et al. (2014), an alternative option would be to use the HV survey estimates to construct the fixed benchmarking weights for yield (R11.S), but they do not satisfy the acreage benchmarking constraint, see Remark for (R11). For this study, the official HV estimates may also be used to construct the fixed benchmarking weights for yield (R11.O), since the study year is 2014 and official estimates are already available. Empirical results show that the yield point estimates are similar for the three cases when harvested acreage estimates are treated as fixed, available from the model (R11), from the survey (R11.S) or

Table 2: Median Absolute Relative Differences of **Agricultural Statistics District**-Level Estimates
2014 **Corn and Soybeans** in Illinois, Indiana and Iowa

			$median_{ij} \left\lvert \frac{MERB_{ij} - ASB_{ij}}{ASB_{ij}} \right\rvert$		$median_{ij} \left\lvert \frac{MERB_{ij} - ASB_{ij}}{MERBSE_{ij}} \right\rvert$		$median_{ij} \left\lvert \frac{MERBSE_{ij} - CAPSSE_{ij}}{CAPSSE_{ij}} \right\rvert$	
	State	Estimator	PD	YD	PD	YD	PD	YD
Corn	IL	R11	1.40%	0.25%	2.48	0.31	89.18%	13.61%
		R12	1.40%	0.25%	0.44	0.31	49.33%	12.77%
	IN	R11	2.66%	0.27%	3.77	0.32	88.60%	23.93%
		R12	2.66%	0.27%	0.61	0.32	24.20%	21.31%
	IA	R11	3.54%	0.20%	6.97	0.33	87.60%	9.08%
		R12	3.54%	0.20%	1.04	0.32	13.69%	7.68%
Soybeans	IL	R11	1.37%	0.31%	1.68	0.49	86.59%	13.60%
		R12	1.37%	0.31%	0.48	0.48	33.86%	12.76%
	IN	R11	3.94%	0.44%	4.09	0.67	88.21%	28.98%
		R12	3.94%	0.44%	0.64	0.64	26.19%	26.86%
	IA	R11	4.35%	0.29%	7.95	0.44	86.69%	20.42%
		R12	4.35%	0.29%	1.27	0.43	13.88%	19.48%

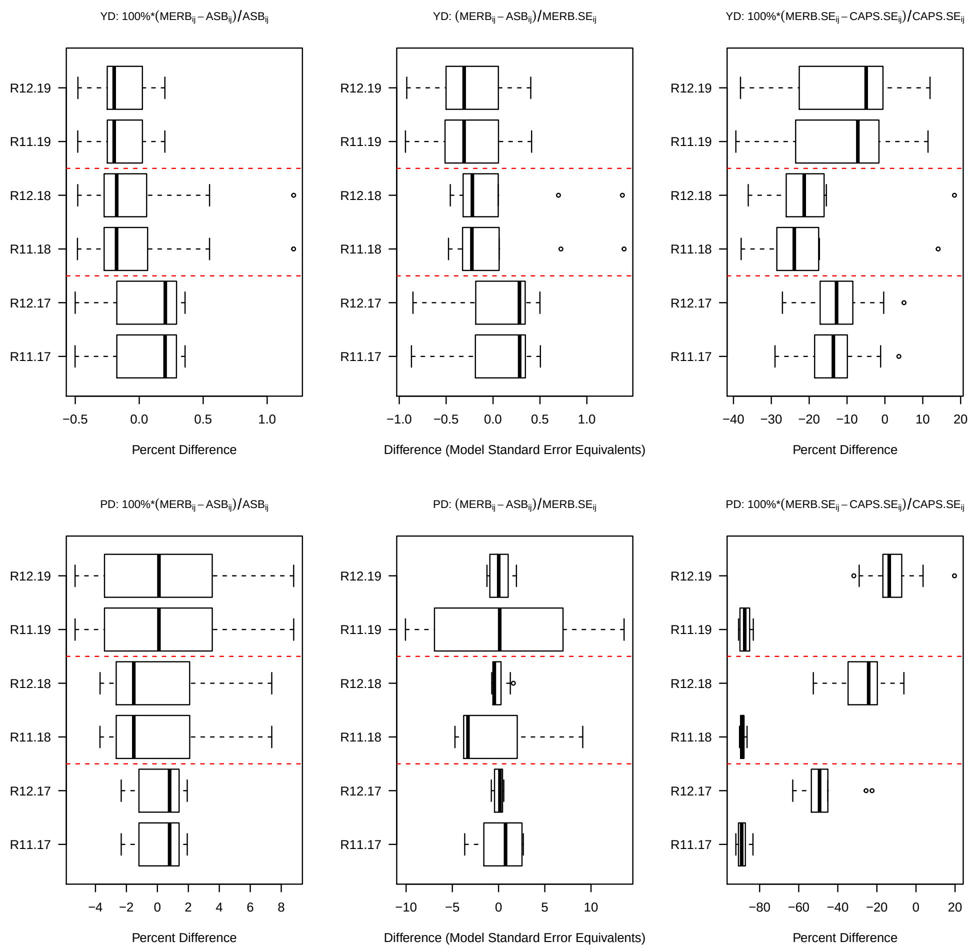
from the published source (R11.O). However, the median relative differences between the production estimates derived using R11.S and the official production estimates are approximately four times larger than the corresponding median relative differences between the production estimates derived using R11 and the official production estimates. Also, the median relative differences between production estimates derived using R11.O and the official production values are approximately seven times smaller than the corresponding median relative differences between the R11 production estimates and the official production values.

Alternatively, model-based estimates may be constructed for the totals, using survey totals estimates and auxiliary data, and ratio estimates may be derived using a ratio estimator. Hence, no additional benchmarking step is needed for the ratio estimates. Similar to the model fit and estimation for acreage, presented in Section 3.3.1, we fit model (15, 16) with $c_{ij} = n_{ij}$ to the subarea-level production survey estimates, and constructed producton model-based estimates for the application study. For this approach, four cases are compared. In the first application, iterates for the ratio are constructed using the MCMC iterates for the numerator and for the denominator, in two different ways: in the first case (R211), the iterates for the ratios are based on the denominator estimates, treated as fixed and known, and in the second case (R212), the iterates for the ratios are based on the MCMC iterates for the denominator. In the second application, ratio estimates are constructed using a Taylor series expansion, and two cases (R221, R222) of constructing covariance estimates between the numerator and the denominator are considered: using the MCMC iterates for the two totals or using the value from the survey summary.

Compared to the derived R11 and R12, the differences between the model-based PD estimates R211, R212, R221, R222 and corresponding ASB estimates are larger; however, most are within one standard error. Under this alternative approach, the median estimated error in the survey PD estimates is reduced $11 - 36\%$, as a result of the model's strength of borrowing information across counties and ASDs and from auxiliary data.

Compared to the model-based estimates R11 and R12, the differences between the derived YD estimates R211, R212, R221, R222 and the corresponding ASB estimates are larger; however, most are within 1.48 standard errors, likely a result of poor variance esti-

Figure 3: **Agricultural Statistics District**-level Model-based Estimates versus ASB Estimates
2014 **Corn** in Illinois, Indiana and Iowa



The box plots illustrate the relationship between the ASD-level direct survey estimates, the model-based estimates (R11, R12) and the ASB estimates, for production and yield. The model-based estimates are ratio benchmarked using the methods described in Section 2. The three horizontal sets of box plots correspond to the three states, denoted by their FIPS codes (17 for Illinois, 18 for Indiana and 19 for Iowa). Results for soybeans are similar.

mation since R211, R212, R221 do not take into account the positive correlation between the two totals and R222 uses a sample-based estimated correlation. The covariance estimator using the MCMC (R221) is weak (a bivariate model may improve it), but overall it performs comparable to R211 and R212. Fixing the correlation to its survey value improves the Taylor variance approximation for the ratio, with greater improvement for soybeans than for corn; the median difference between the R222 model-based standard error and the survey standard error is less than one half for corn and less than one forth for soybeans, compared to R211, R212 and R221. The Taylor variance approximation needs further attention.

The alternative constructions R211, R212, R221, R222 have the potential to increase the number of county-level yield estimates, since, for some states and commodities, the county-level sample size may be lower for yield, than for acreage and production. From the set of six different constructions considered in this manuscript, R221 and R222 satisfy both the benchmarking constraints and the relationship between the three parameters of interest. The only challenge is in estimating the covariance between the numerator (production) and the denominator (harvested acreage); in R222 we used the estimated correlation (treated as fixed) available from the survey summary, and based on the common number of records available for the three parameters. Bivariate small area modeling of acreage and production survey estimates would provide reliable estimates for acreage and production, as well as estimated variance-covariance matrix. Yield estimates using R222 could then be constructed using the MCMC iterates from the bivariate model; research left for future investigation.

4. Discussion

NASS's interest in producing reliable end-of-season yield estimates and associated standard errors, subject to constraints at multiple aggregation levels, has motivated the research presented in this paper. The problem of estimating and benchmarking three related quantities of interest lead to the investigation of the effect of benchmarking constraints for ratios. We presented different methods of constructing model-based estimates for two totals and their ratio at lower-level areas that aggregate to fixed values at upper-levels. In this paper, we introduce and compare methods of applying the benchmarking adjustments to model-based estimates at multiple stages. We chose to illustrate the methods using the ratio benchmarking adjustment, one of the two classic benchmarking adjustments: ratio and difference. The reason we did not pick the difference adjustment is that it may lead to negative estimates for the ratio of positive totals.

The (R11) application of benchmarking adjustment to ratio estimates, where the estimates for the denominator are treated as fixed and known, is common practice at NASS for projects where only one quantity is of interest (such as yield forecasting or cash rental rate estimation). We showed that the (R11) estimates are conditional on the denominator estimates, resulting in an underestimated variance for production estimates. As a solution to produce reliable numerator estimates, we introduced method (R12). For the application study, the construction (R12) provides a valuable set of estimates for the triplet, the inconsistencies in the ratio identity may be avoided with a simple rounding. Moreover, the estimated CVs for the county-level yield (R12) estimates are approximately $80 - 90\%$ of the corresponding estimated CVs for the county-level survey estimates and the estimated CVs for the county-level production (R12) estimates are approximately $40 - 45\%$ of the corresponding estimated CVs for the county-level survey estimates.

We presented harvested acreage, production and yield estimates for 2014 end-of-season corn and soybeans in three selected states. Posterior summaries are available for the county-level estimates and ASD-level estimates, for the triplet that closely satisfies the benchmarking constraints at both levels. Extensions of the methods, to produce posterior summaries for any intermediate aggregation level between county-level and state-level, such as groups of counties or groups of ASDs, is straightforward. In this paper, we presented posterior means and posterior variances and compared the results to the survey estimates and to the official estimates. We showed consistency between the *modeled* YD estimates (R11, R12) with survey and official, but additional results show inconsistencies between the *de-*

rived YD estimates (R211, R212, R221, R222) from modeled production and acreage. The model-based estimates have smaller variances than the survey estimates for the three quantities of interest.

Due to construction, the point estimates for the ratio derived under (R211), (R221) and (R222) are the same, but the measures of uncertainty differ. Only the constructions of ratio estimates given in (R11), (R211), (R221) and (R222) satisfy the benchmarking constraints and the ratio definition exactly. In (R11) and (R211), the estimates for the ratio are conditional on the estimates for the denominator. In (R221) and (R222), the estimate for the variance of the ratio depends on the estimate for the covariance between the numerator and the denominator. The most robust approach to construct estimates for the three parameters would be to model $T1$ and $T2$ jointly, and then construct R as the ratio, with associated variance derived using a Taylor series expansion for the ratio; an estimated covariance term would be available from the estimation based on the joint model for $T1$ and $T2$. In an operation environment, (R11) or (R12) may be used, with a caution that (R11) is conditional on the denominator. If the user is concerned with production and yield estimation only, assuming that good harvested acreage values are already available, then the (R11.O) construction may be used. The (R211), (R212), (R221) and (R222) constructions may only be used when the variance estimation is improved.

If the survey estimate for the ratio is not available for a subarea j, then the model-based estimate for the ratio is not constructed for the subarea j. As a result, under (R11) and (R12), the model-based numerator estimate for subarea j is not constructed, even if the denominator estimate is available. On the other hand, when the survey estimates for the numerator and denominator are available for a subarea j, model-based estimates for the numerator and denominator are constructed and, under (R211), (R212), (R221), (R222), ratio estimates are derived, even if the survey ratio estimate is not available. Hence, the (R11) and (R12) constructions result in a smaller number of estimates for the ratio, than the (R211), (R212), (R221), (R222) constructions do.

The emerging remote sensing estimation procedures provide good sources of yield estimates, available for selected commodities and for selected states. See, for example, Dave Johnson (2014). However, the remote sensing estimates are subject to coverage errors due to the environment, are not benchmarked to the state-level values, and measures of uncertainty associated with the estimates are not available. Applying a benchmarking adjustment to such estimates may or may not fall under one of the methods illustrated in this manuscript, and it is left for future investigation. Our preliminary results show that the remote sensing estimates are closer to the derived yield estimates in (R211, R212, R221, R222) than to the model-based estimates (R11, R12).

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Appendix A

A1. Model T1 and R. Fixed Benchmarking Weights.

For a ratio benchmarking adjustment, the proof is as follows

$$
\begin{aligned}
\sum_j \hat{\theta}_j^{T1,B} &= \sum_j \hat{\theta}_j^{T2,B} \hat{\theta}_j^{R,B}, \\
&= \sum_j \hat{\theta}_j^{T2,B} K^{-1} \sum_k \theta_{jk}^{R}, \text{ by } (6) \\
&= \sum_j \hat{\theta}_j^{T2,B} K^{-1} \sum_k \theta_{jk}^{R,noadj} a_R \left(\sum_j w_{jk}^{R} \theta_{jk}^{R,noadj} \right)^{-1}, \text{ by } (4) \\
&= a_R K^{-1} \sum_k \sum_j \hat{\theta}_j^{T2,B} \theta_{jk}^{R,noadj} \left(\sum_j w_{jk}^{R} \theta_{jk}^{R,noadj} \right)^{-1}, \text{ since } a_R \text{ is constant} \\
&= a_R K^{-1} \sum_k \sum_j \hat{\theta}_j^{T2,B} \theta_{jk}^{R,noadj} \left(\sum_j a_{T2}^{-1} \hat{\theta}_j^{T2,B} \theta_{jk}^{R,noadj} \right)^{-1}, \text{ by definition of } w_{jk}^{R} \\
&= a_R a_{T2} K^{-1} \sum_k (\sum_i \hat{\theta}_j^{T2,B} \theta_{jk}^{R,noadj}) \left(\sum_j \hat{\theta}_j^{T2,B} \theta_{jk}^{R,noadj} \right)^{-1}, \text{ since } a_{T2} \text{ is constant} \\
&= a_R a_{T2} = a_{T1}.
\end{aligned}
$$

Similarly, the proof for area-level equality constraints follows immediately,

$$
\begin{aligned}
\sum_i \hat{\theta}_i^{T1,B} &= \sum_i \hat{\theta}_i^{T2,B} \hat{\theta}_i^{R,B}, \\
&= \sum_i K^{-1} \sum_k \hat{\theta}_i^{T2,B} \theta_{ik}^{R}, \text{ by } (6) \\
&= \sum_i K^{-1} \sum_k a_{T2} w_i^{R} \theta_{ik}^{R}, \text{ by definition of } w_{ik}^{R} \\
&= a_{T2} a_R = a_{T1}.
\end{aligned}
$$

The ratio estimator for the set of all subareas is the weighted sum of the subarea-level ratio estimators,

$$
\begin{aligned}
\sum_j w_j^{R} \hat{\theta}_j^{R,B} &= \sum_j w_j^{R} K^{-1} \sum_k \theta_{jk}^{R}, \text{ by } (6) \\
&= K^{-1} \sum_k \sum_j w_j^{R} \theta_{jk}^{R}, \text{exchanging summations} \\
&= K^{-1} \sum_k \sum_j w_j^{R} \theta_{jk}^{R,noadj} a_R \left(\sum_j w_j^{R} \theta_{jk}^{R,noadj} \right)^{-1}, \text{ by } (4) \\
&= K^{-1} \sum_k a_R, \text{ since } a_R \text{ is constant} \\
&= a_R.
\end{aligned}
$$

A2. Model T1 and R. Random Benchmarking Weights.

Note that, from the derivation of production total estimates, the benchmarking weights for the two totals are equal. Hence, $w_j^{T1} = w_j^{T2} = w_{jk}^{T2}$, for all $k = 1, ..., K$.

For a ratio benchmarking adjustment, the proof is as follows

$$\begin{aligned}
\sum_j w_j^{T1}\hat{\theta}_j^{T1,B} &= \sum_j w_j^{T1} K^{-1} \sum_k \theta_{jk}^{T1} \\
&= \sum_j w_j^{T1} K^{-1} \sum_k \theta_{jk}^{T2}\theta_{jk}^{R}, \text{ by } (13) \\
&= \sum_j w_j^{T1} K^{-1} \sum_k \sum_j \theta_{jk}^{T2}\theta_{jk}^{R,noadj} a_R \left(\sum_j w_{jk}^{R}\theta_{jk}^{R,noadj}\right)^{-1}, \text{ by } (4) \\
&= a_R K^{-1} \sum_k \sum_j w_j^{T1} a_{T2} w_{jk}^{R} (w_{jk}^{T2})^{-1} \theta_{jk}^{R,noadj} \left(\sum_j w_{jk}^{R}\theta_{jk}^{R,noadj}\right)^{-1}, \\
&\quad \text{since } a_R \text{ is constant and by definition of } w_{jk}^{R} \\
&= a_R a_{T2} K^{-1} \sum_k (\sum_j w_j^{T1} (w_{jk}^{T2})^{-1} w_{jk}^{R}\theta_{jk}^{R,noadj}) \left(\sum_j w_{jk}^{R}\theta_{jk}^{R,noadj}\right)^{-1}, \\
&\quad \text{since } a_{T2} \text{ is constant} \\
&= a_R a_{T2} K^{-1} \sum_k (\sum_j w_{jk}^{R}\theta_{jk}^{R,noadj}) \left(\sum_j w_{jk}^{R}\theta_{jk}^{R,noadj}\right)^{-1}, \text{ since } w_j^{T1} = w_{jk}^{T2} \\
&\quad , \text{for all } k = 1, ..., K \\
&= a_R a_{T2} = a_{T1}.
\end{aligned}$$

The ratio estimator for the set of all subareas is the weighted sum of the subarea-level ratio estimators,

$$\begin{aligned}
\sum_j w_j^{R}\hat{\theta}_j^{R,B} &= \sum_j w_j^{R} K^{-1} \sum_k \theta_{jk}^{R}, \text{ by } (8) \\
&= \sum_j (K^{-1} \sum_{k'} w_{jk'}^{R})(K^{-1} \sum_k \theta_{jk}^{R}), \text{ since } K^{-1} \sum_{k'} w_{jk'}^{R} = \frac{\hat{\theta}_j^{T2,B}}{a_{T2}} = w_j^{R} \\
&\neq K^{-2} \sum_k \sum_k \sum_j w_{jk}^{R}\theta_{jk}^{R} \\
&= K^{-2} \sum_k \sum_k a_R = a_R.
\end{aligned}$$

The subarea-level numerator total estimator is

$$\begin{aligned}
\hat{\theta}_j^{T1,B} = K^{-1} \sum_k \theta_{jk}^{T1} &= K^{-1} \sum_k \theta_{jk}^{T2}\theta_{jk}^{R}, \text{ by } (13) \\
&\neq \left(K^{-1} \sum_{k'} \theta_{jk'}^{T2}\right) \left(K^{-1} \sum_k \theta_{jk}^{R}\right).
\end{aligned}$$