The Two Sample Problem

Alan H. Dorfman
U.S. Bureau of Labor Statistics, 2 Massachusetts Ave, N.E., Room 1950, Washington, D.C. 20212
dorfman.alan@bls.gov

Abstract

It sometimes happens that two separate samples from a population, having perhaps quite distinct designs and mode of sampling, contribute information on the same variable of interest, and it becomes an important question how to combine the data from the two samples. An example is the Occupational Employment Statistics survey (OES) and the National Compensation Survey (NCS), carried out by the Bureau of Labor Statistics, both contributing information on occupational wages. We discuss some new options for combining data from two samples and achieving unified estimation.

Key Words: Missing Information Principle, Post-stratification, Pseudo-likelihood

1. Introduction – The Question of Combining Two Samples

Suppose two distinct surveys gather related information on a single population *U*. How best to combine data from the two surveys to yield a single set of estimates?

This situation is fairly common. For example, the National health Interview Survey (NHIS) and the Behavioral Risk Factor Surveillance System (BRFSS) both collect data on risk factors for serious illness. The Occupation Employment Statistics program (OES) and the National Compensation Survey (NCS) both collect data on occupational employment and wages.

The typical approach is to get separate estimates of the target parameter from the two surveys and weight them together with weights the inverse of their estimated variances. For example, see (Merkouris 2004) and the references therein.

Another possibility, however, is to combine the two data sets into a single data set and "weight up" appropriately. This can be awkward in the standard design based framework, but nevertheless can pay dividends.

In this paper we consider a simple example where this turns out to be the case. This example is a simplified version of the OES/NCS situation, and was constructed in the hopes of shedding light on what might best work in unifying those two surveys.

2. Study Example

Consider a population *U* of *N* establishments *e*, each with O occupations *o*. Two samples of establishments, s_1 of size n_1 and s_2 of size n_2, with data collected for all occupations in s_1 but only for a sub-sample of *k* occupations in s_2. Let E_{oe} be the employment in *o* in *e*, and $E_{+e} = \sum_o E_{oe}$ the employment in *e*. The goal is to estimate the proportion $p_o \equiv \dfrac{E_{o+}}{E_{++}}$ of employment in each of the O occupations. Here $E_{o+} = \sum_e E_{oe}$ is total employment in occupation *o* and $E_{++} = \sum_o E_{o+}$ is total employment.

In this context, it is natural to consider a Multinomial model, where $\{E_{oe}\} \sim Multinom\left(E_{+e}, \{p_o\}\right)$. Then the population log likelihood would be $\log L = \sum_e \left(\sum_{o=1}^{O-1} E_{oe} \log p_o + E_{Oe} \log p_O \right)$, with score functions

$$\frac{\partial L}{\partial p_o} = \sum_e \left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \right) = \frac{E_{o+}}{p_o} - \frac{E_{O+}}{p_O}, \quad o = 1, 2, ..., O - 1. \qquad (1)$$

Thus, if we had data for whole population, we would set the score function to zero and solve, getting estimates

$$\hat{p}_o = \frac{E_{o+}}{E_{++}}, \quad o = 1, 2, ..., O,$$

so that, not surprisingly, the estimates would equal the targets:, $\hat{p}_o = p_o$. But, of course, we do not have the population data, only what has been collected in the two samples, plus some possible population auxiliary information. We shall assume that overall establishment employment E_e is available for all establishments, in and out of sample.

2.1 Pseudo-likelihood

One commonly accepted way to handle maximum likelihood problems in the survey context is to use *Pseudo-likelihood* . Replace population sums in the population score function by sample estimates of those sums, using the inverse of selection probabilities, the design weights w_e:

$$\frac{\partial L}{\partial p_o} = \sum_{e \in s} \left(\frac{w_e E_{oe}}{p_o} - \frac{w_e E_{Oe}}{p_O} \right) = \frac{\hat{E}_{o+}}{p_o} - \frac{\hat{E}_{O+}}{p_O}, \quad o = 1, 2, ..., O - 1. \qquad (2)$$

Setting these pseudo-likelihood score functions to zero and solving leads to $\hat{p}_o = \dfrac{\hat{E}_{o+}}{\hat{E}_{++}}$.

If, for example, on sample 1, units are selected *pps*(E_e), we could get an estimate from that sample using

$$\frac{\partial \hat{L}}{\partial p_o} = \sum_{e \in s1} \pi_e^{-1} \left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \right) = \sum_{e \in s1} \frac{E_{..}}{n_1 E_e} \left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \right) = 0.$$

Solving gives

$\hat{p}_o^{\{1\}} = \dfrac{1}{n_1} \displaystyle\sum_{e \in s1} \dfrac{E_{oe}}{E_e}$, the pseudo-likelihood estimator for sample 1.

For combining the surveys we could, similarly, get an estimate from second sample, and then weight these together based on our estimate of the variances of the two estimates. This would be in keeping with the approach usually suggested. Can we use pseudo-likelihood on a combined version of the data from the two surveys?

2.2 Dual sample pseudo-likelihood estimator

Suppose the two samples are independent, with inclusion probabilities $\pi_{oe}^{(1)}, \pi_{oe}^{(2)}$.

Then the probability that *oe* gets into the combined sample is $\pi_{oe}^{(dual)} = \pi_{oe}^{(1)} + \pi_{oe}^{(2)} - \pi_{oe}^{(1)} \pi_{oe}^{(2)}$.

Let $w_{oe}^{(dual)} = 1/\pi_{oe}^{(dual)}$. Then we can use these weights in (1) to get a pseudo-likelihood estimator

$$\hat{p}_o^{\{dual\}} = \frac{\sum\limits_{e \in s, comb} w_{oe}^{(dual)} E_{oe}}{\sum\limits_o \sum\limits_{e \in s, comb} w_{oe}^{(dual)} E_{oe}}$$

This idea of using a combined inclusion probability underlies a lot of the work on the problem of dual frame estimation.

2.3 Missing Information Principle

Another approach, quite different from pseudo-likelihood, is the application of the Missing Information Principle (Breckling, et al. 1994). This gives the *actual* maximum likelihood estimates of the targets, given all the data (from each sample and the auxiliary data) and the truth of the model, in this case the simple multinomial model. Here is a sketch of the principles involved:

Suppose $\frac{\partial \log L(x;\omega)}{\partial \omega}$ is the score function if we have "full information" *x*, and let *D* represent the portion of *x* for which we actually have data, then the score function given that data can be written as the expectation of the original score function conditional on *D*,

$$\frac{\partial \log L(D;\omega)}{\partial \omega} = E_{\omega}\left(\frac{\partial \log L(x;\omega)}{\partial \omega} \mid D\right).$$

This is a strict application of maximum likelihood and depends strongly on the truth of the (multinomial) model. Note that there is no use of inclusion probabilities. We spell out how this works in the present example:

Units *e* fall into three categories, depending on what we know about them, as described in Table 1.

Table 1: Classification of units ("establishments") in the Example

	o = 1	o = 2	...	o = O	
s_1	E_{1e}	E_{2e}		E_{Oe}	E_{+e}
$s_{2\backslash 1}$	*** or E_{1e}	*** or E_{2e}		*** or E_{Oe}	E_{+e}
$r = U - (s_1 \cup s_2)$	***	***		***	E_{+e}

In the table, "***" means "data is missing". Thus, the units that are in neither sample, are always missing employment on occupations, those in s_1 always have occupational employment (whether or not they are also in s_2), and those in s_2 but not in s_1 will, for a given occupation, possibly have the occupations employment, but possibly not.

Thus the expectation of the population score function (1), conditional on the sample data, takes the form

$$E\left(\frac{\partial L}{\partial p_o} \mid D\right) = \sum_{e \in s1}\left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O}\right) + \sum_{e \in s2\backslash 1} E\left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \mid D\right)$$

$$+\sum_{e \in r} E\left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \mid D\right)$$

The last term is zero:

$$E\left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \mid E_{+e}\right) = \frac{E_{+e}\, p_o}{p_o} - \frac{E_{+e}\, p_O}{p_O} = 0$$

The second term is messy: $E\left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \mid D\right) = E\left(\frac{E_{oe}}{p_o} - \frac{E_{Oe}}{p_O} \mid E_{+e}, \left\{E_{o'e}\right\}_{o' \in s_e}\right)$,

where s_e is the sample of occupation in establishment *e* .

For example, if both o and O are in s_e then the term looks like terms in the s_1 sum. If o is not in s_e, then, letting $E_{+e}^{2\backslash 1} = E_{+e} - \sum_{o'\in s_e} E_{o'e}$ and $p_o^{s_e} = \dfrac{p_o}{1-\sum_{o'\in s_e} p_{o'}}$, we have $E\left(E_{oe} \mid E_{+e}, \left\{E_{o'e}\right\}_{o'\in s_e}\right) = E_{+e}^{2\backslash 1} p_o^{s_e}$

The net result is the sample score function

$$E\left(\frac{\partial L}{\partial p_o} \mid D\right) = \frac{1}{p_o}\left\{\sum_{e\in s1} E_{oe} + \sum_{e\in s2\backslash 1, o\in s_e} E_{oe} + \sum_{e\in s2\backslash 1, o\notin s_e} E_{+e}^{2|1} p_o^{s_e}\right\} -$$

$$\frac{1}{p_O}\left\{\sum_{e\in s1} E_{Oe} + \sum_{e\in s2\backslash 1, O\in s_e} E_{Oe} + \sum_{e\in s2\backslash 1, O\notin s_e} E_{+e}^{2|1} p_O^{s_e}\right\}.$$

Setting this equal to zero to get estimates of the p_o leads to

$$\hat{p}_o = \frac{\tilde{E}_o}{\sum_{o'=1}^{O} \tilde{E}_{o'}}, \qquad (3)$$

where $\tilde{E}_o \equiv \sum_{e\in s1} E_{oe} + \sum_{e\in s2\backslash 1, o\in s_e} E_{oe} + \sum_{e\in s2\backslash 1, o\notin s_e} E_{+e}^{2|1} p_o^{s_e}$. This looks simple, but actually is not, since the p_o we are solving for are components of the terms on the right side of (3). However, we can solve by iteration, plugging in initial values for p_o into the expressions for the $p_o^{s_e}$, to calculate the right side of (3), giving to get new values $\hat{p}_o$. These new values can in turn to be plugged in, and the process repeated.
This works well.

2.4 Iterated post-stratified estimator

What if, in estimating, we do not wish to assume the multinomial model? One approach involves the well known sampling technique of *poststratification*. In post-stratification, the population is divided into strata *after* the sample data are collected. Estimation proceeds as if these were original design strata.

If the stratification variable ***x*** is continuous, then it often seems to be arbitrary where stratum boundaries are drawn. A way around this arbitrariness, is to do several poststratifications, using systematically different boundaries. In the *iterated post-stratified estimator* several post-stratifications are constructed, each leading to a single estimate. The final estimate is taken as a simple average of these.

For the simplified NCS-OES like population with two samples, this would be accomplished by the following sequence:

(1) we take the stratification variable to be E_e.

(2) For each occupation o, let $s(o)$ be the set of e for which the combined sample data provide E_{oe} .
(that is, all sample 1 e's and sample 2 e's, for which E_{oe} is sub-sampled) Let n_o be the size of this set.

(3) Let k be number of post-stratifications. This is at the discretion of the analyst.
How to choose a best k is an open question.

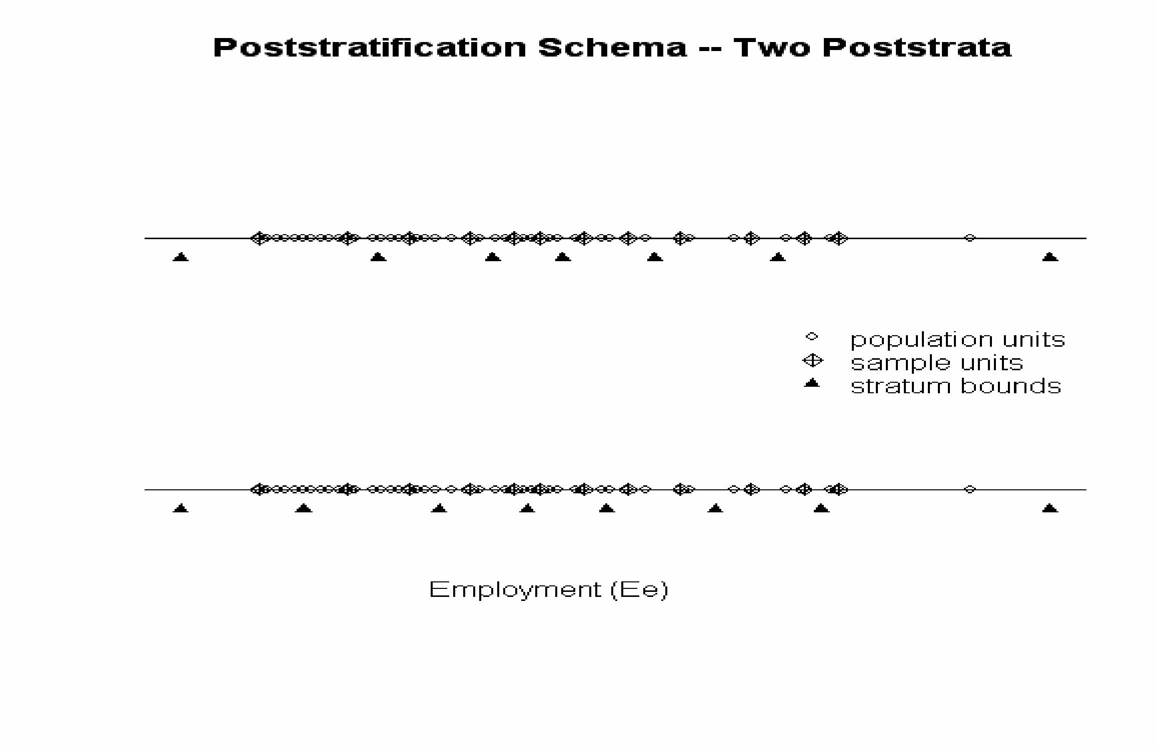
(4) for the lower bound of lowest stratum of any of the poststratifications, we can take any E^* that is less than all E_e in the population. Likewise, the upper bound of highest stratum is any E^{**} greater than all E_e.

(5) In first post-stratification take intermediate bounds of post-strata halfway between values of E_e at jkth and jkth + 1 sample points in $s(o)$, for j = 1,…,[n_o/k].

For the second post-stratification, take intermediate bounds between the jkth +1 and jkth + 2 sample points in $s(o)$, etc.

This puts $n_{h(o,i)} = k$ sample units in each post-strata $h(o,i)$ of ith poststratification, except possibly in the boundary strata

The number of population units $N_{h(o,i)}$ with E_e within the stratum boundaries will in general vary from stratum to stratum. The following Figure depicts schematically the choice of stratum boundaries for the several poststrata for a given occupation.



(6) Then estimate the employment in o in the stratum by $\hat{E}_{o,h(o,i))} = \frac{N_{h(o,i)}}{n_{h(o,i)}} \sum_{e \in h(o,i) \cap s(o)} E_{oe}$, and get overall

$$\hat{E}_{o(i)} = \sum_h \hat{E}_{o,h(o,i))} ,$$

the estimated employment in o, based on the ith postratification.

(7) Take the average of these as the overall estimate of employment in the occupation:
$\hat{E}_o = \sum_i \hat{E}_{o(i)} \Big/ k$.

Note: The implicit weight w_{oe} on sample value E_{oe} is

$$w_{oe} = \frac{1}{k} \sum_{i=1}^{k} I\left(e \in h(o,i) \cap s(o)\right) \frac{N_{h(o,i)}}{n_{h(o,i)}} .$$

(7) Repeat for the other occupations o'.

(8) Take $\hat{p}_o = \hat{E}_o \Big/ \sum_{o'} \hat{E}_{o'}$ as the iterated post-stratified estimator of p_o.

3. Simulation Studies

3.1 Simulation Study 1

3.1.1 Population 1 - Description

Using a lognormal random generator, we created 500 "establishments" with sizes E_e ranging from 12 to 385. Their median size was 97.

Each establishment's employment was divided among 5 "occupations" using a multinomial distribution. In particular, occupational employments in *e* were generated by

$$Multinom(E_e,\ \mathbf{p} = (.05,.1,.23,.27,.35))$$

The corresponding proportion of employment in the population turned out to be

$$\mathbf{p}_U = (.051\ 0.098\ 0.230\ 0.271\ 0.350)$$

The goal is to estimate $\mathbf{p}_U$, based on data from two samples.

3.1.2 Sampling Methodology and Estimators

500 pairs of samples were taken:

Sample 1 $pps(E_e)$ n_1 = 50 – with a census of *e*'s Occupations
Sample 2 $pps(E_e)$ n_2 = 30 – in this case a sub-sample s_e of 2 occupations was sampled $ppswr(E_{oe})$.

We calculated the following estimators:

(i) the *pseudo-likelihood estimator* $\hat{p}_o^{\{1\}} = \frac{1}{n_1}\sum_{e\in s1}\frac{E_{oe}}{E_e}$

(ii) the *pwr estimator* on sample 2,
based on *with replacement* sampling of occupations:

$$\hat{p}_o^{\{2\}} = \frac{1}{2n_2}\sum_{e\in s2}\sum_{k=1}^{2} I_{ek}(o),$$

where $I_{ek}(o)$ = 1, if *o* selected at *k*th selection of occupation in *e*,

and $I_{ek}(o)$ = 0, otherwise.

(iii) A weighted combination of the above estimators:

Let $w_i = \text{var}^{-1}\left(\hat{p}_o^{\{i\}}\right)$, estimated empirically over the 500 samples (actually used median of these across the 5 occupations)

$$\hat{p}_o^{\{\theta\}} = \frac{w_1\hat{p}_o^{\{1\}} + w_2\hat{p}_o^{\{2\}}}{w_1 + w_2}$$

(iv) $\hat{p}_o^{\{mip\}}$, the MIP max'm likelihood estimator

(v) $\hat{p}_o^{\{1ml\}}$, the MIP max'm likelihood estimator, using only data from sample 1

(vi) $\hat{p}_o^{\{dual\}}$ the *dual* estimator, i.e. the pseudo-likelihood estimator, using two-sample π's

(vii) $\hat{p}_o^{\{iter.ps\}}$ the iterated post-stratified estimator

3.1.3 Results

Relative biases and root mean square errors are given in Table 2. There is not too much to distinguish among the estimators with respect to Bias.

With respect to root mean square error: There is little difference between *pseudo -1* (the pseudo-likelihood estimator based on just sample 1) and the combined estimator. In other words, combining the *results* of estimation from the two surveys does not improve on just using the big survey. The *mip* is distinctly best, and *does* improve on *mip* -1. That is, using this estimator—which combines the *data* from the two surveys

Table 2: Simulation Results – Population 1

	Relative Bias X 1000					**Root Mean Square Error X 1000**				
	Occ1	*Occ2*	*Occ3*	*Occ4*	*Occ5*	*Occ1*	*Occ2*	*Occ3*	*Occ4*	*Occ5*
pseudo -1	-2.5	1.4	0.7	-0.3	-0.2	2.7	3.7	5.5	5.6	6.2
pwr -2	-11.8	8.8	4.2	3.7	-6.4	30.0	38.2	56.2	56.4	63.4
combined	-2.5	1.6	0.7	-0.3	-0.3	2.7	3.7	5.5	5.6	6.2
mip	7.5	-11.2	1.1	4.2	-1.9	**2.4**	**3.1**	**3.9**	**4.2**	**4.7**
mip - 1	7.9	-11.2	1.4	3.6	-1.7	2.4	3.3	4.7	4.9	5.5
dual	-1.9	3.1	-0.8	1.2	-1.0	3.3	5.1	10.8	11.5	13.5
iterated ps	2.0	5.0	1.7	-0.9	-2.1	2.9	3.8	5.2	5.1	5.8

-- *does* capitalize on the extra bit of information in sample 2. The iterated post-stratified estimator is here sometimes better, sometimes marginally worse than the pseudo-likelihood estimator. The dual estimator based on overall inclusion probabilities is considerably worse than the other estimators relying on both samples.

3.2 Simulation Study 2

3.2.1 The Question: What if the Multinomial Model is Wrong?

Consider the following Population: Workers in *e* with large E_e have probability p_o^L of being in *o*. Workers in *e* with small E_e , have probability p_o^S of being in *o.* Suppose $p_o^L < p_o^S$ and we aim to estimate $p_o = \sum_U E_{oe} \Big/ \sum_U E_e$ based on a single sample, where sampling was *pps*, with size measure $E_{e.}$ In this, single sample case, the pseudo-likelihood and the *mip* estimators take the forms $\hat{p}_o^{\{1\}} = \dfrac{1}{n_1} \sum_{e \in s(1)} \dfrac{E_{oe}}{E_e}$

$\hat{p}_o^{\{1ml\}} = \dfrac{\sum_{e \in s(1)} E_{oe}}{\sum_{e \in s(1)} E_e}$, respectively. On the one hand, $\hat{p}_o^{\{1\}}$ is design-unbiased for p_o . But in the *mip* $\hat{p}_o^{\{1;MIP\}}$, large E_{oe} dominate sums in numerator and denominator. Under the assumptions above, one can anticipate that $\hat{p}_o^{\{1;MIP\}} \approx p_o^L < p_o$, so that the *mip* will be seriously biased down. Thus the success of the *mip* depends on getting the model right.

3.2.2 Population 2 - Description

For Population 2 we use the same establishment employments E_e as Pop 1, and same first stage samples. But E_{oe} has distinct model for 250 smallest and 250 largest E_e, as in Table 3. Second stage sampling is carried out as it was in Simulation Study 1.

Table 3: Probabilties in Population 2 of Allocation of Employment to Different occupations

Size\occupation	1	2	3	4	5
Small Establishments	0.05	0.10	0.23	0.27	0.35
Large Establishments	0.35	0.27	0.23	0.10	0.05

For occupation 4 and 5, MIP based on overall multinomial model should be biased low and for 1 and 2, high.

Table 4: Overall Population 2 Proportions to be Estimated

Occupation	1	2	3	4	5
Over all	0.25	0.21	0.23	0.16	0.15

3.2.3Simulation Study 2 Results

Relative biases and root mean square errors are given in Table 5. The relative biases *do* show a striking differences between the *mip* and the other estimators, as was predicted:

Table 5: Simulation Results for Population 2

	Relative Bias X 1000					**Root Mean Square Error X 1000**				
	Occ1	Occ2	Occ3	Occ4	Occ5	Occ1	Occ2	Occ3	Occ4	Occ5
pseudo -1	5.2	2.6	-1.1	-2.7	-7.8	20.7	11.7	5.7	12.2	20.9
pwr -2	-11.0	24.3	-10.5	21.6	-22.4	61.5	54.8	54.4	49.7	47.2
combined	3.5	3.8	-1.2	-0.8	-10.3	19.6	11.3	5.7	11.4	19.0
mip	*193.9*	*106.4*	*-29.7*	*-166.0*	*-259.9*	50.4	24.0	8.3	26.9	41.1
dual	-0.1	7.8	-2.4	2.8	-10.3	19.1	13.2	10.9	11.6	17.9
iterated ps	-4.1	-4.5	-3.3	5.9	12.3	**6.1**	**4.8**	**5.2**	**5.2**	**6.5**

Its bias leads to the *mip* having large *rmse.* Indeed, the p*seudo-likelihood* estimator based just on sample 1 does much better than *mip* based on both samples. The combined estimator is marginally better than sample 1 *pseudo-likelihood* . The *dual* estimator is sometimes better than sample 1 *pseudo - likelihood* estimator. But the c**lear winner** is the *iterated post-stratified estimator.*

4. Discussion

Success of *mip* depends on model being correct. However, it may be that Population 2 is an extreme case. We might be able in practice to use prior knowledge or diagnostics to handle this problem – this is something to investigate. In neither population, did the design based ways of combining estimates or data yield best results. The iterated post-stratified estimator looks to be a very promising non-parametric way to combine data from two sources.

Acknowledgements

Thanks to Ronald Johnson of BLS for comments improving the presentation of this paper.

References

Breckling, J. U., Chambers, R. L., Dorfman, A. H., Tam, S. M., and Welsh, A. H. (1994), Maximum Likelihood Inference from Sample Survey Data, International Statistical Review, 62, 349-363

Merkouris, T. (2004) Combining Independent Regression Estimates from Multiple Surveys, Journal of the American Statistical Association, 99, 1131-1139

The views expressed are the author's and do not reflect Bureau of Labor Statistics policy.