ON THE VARIABILITY OF ESTIMATES BASED ON PROPENSITY SCORE WEIGHTED DATA FROM WEB PANELS

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1 Introduction

Consider a sample survey of the general population. The survey goal is to estimate some population entity, say, the population mean. Under most standard sampling designs, such as simple random sampling (SI) or stratified SI, this is a straightforward task: suitable formulae are available in any sampling textbook (for instance, Cochran, 1977, or Särndal et al, 1992.) The textbooks, however, rarely offer any advise on how to estimate the mean if a nonprobability procedure is used to select the sample. Then, the design-based theory does not hold, but inference must rely on model assumptions.

Here, we focus on Web surveys, which typically suffer from both a lack of appropriate sampling frames and a low Internet penetration in the general population. In consequence, they often have to rely on volunteer panels. A standard (design-based) estimator of the mean – or some other parameter – may suffer from severe selection bias if applied on the panel data. As a means of avoiding this, a model-based 'propensity score estimator' has been proposed. Under ideal conditions, this estimator may be free from selection bias. A remaining issue, dealt with in this paper, is how to estimate its variance.

1.1 The problem

Our starting-point is a recurrent sample survey consisting of two parts, characterized by the data collection method in use: telephone (T) or Web (W). The Web is the main data collection method, whereas data collection by telephone is a rare event, performed for the sole purpose of aiding the estimation.

The parameter to be estimated is the population mean, $\bar{y}_U = \sum_{k \in U} y_k/N$, where U is the general population (of size N) and y_k is the fixed value on study variable y for individual $k \in U$.

The Web sample s_W is selected from U_W: a subset of U. In practice, we think of s_W as chosen from some volunteer panel of Internet users, possibly created by inviting visitors on popular Internet sites and portals (corresponding to Type 3 in Couper's taxonomy of Web surveys, (Couper, 2000).) The telephone sample s_T, on the other hand, is an SI sample from U (to simplify, we assume that the frame population of the telephone survey coincides exactly with U.) The sizes of s_W, s_T, U_W and U are denoted n_W, n_T, N_W and N, respectively. The size of the total sample $s = s_T \cup s_W$ is $n = n_T + n_W$.

The problem is to estimate the mean of U from s_W, supported by s_T, and to quantify the uncertainty in this estimate.

1.2 Our approach

We deal with the situation described in sec. 1.1 by leaving the finite population framework and look upon y_k ($k \in U$) as a random variable associated with the kth individual (the actual y_k is taken as a

realization of this random variable.) Our viewpoint brings us to a model world sometimes referred to as a *superpopulation model*–see, for instance, Särndal et al (1992, sec. 12.2) or Cassel et al (1977). The random variables $y_1, ..., y_N$ are regarded as independently and identically distributed (iid) with a common mean $E(y_k) = \mu$ and variance $V(y_k) = \sigma^2$ for $k \in U$. From general properties of a random sample (Casella and Berger, 1990, Theorem 5.2.2), the expectation and variance of $\bar{y}_U$ are then given by

$$E(\bar{y}_U) = \mu; \qquad V(\bar{y}_U) = \frac{\sigma^2}{N} \tag{1}$$

In this setting, the estimation problem discussed in sec. 1 is translated into the one of estimating μ and σ^2 from available data. To accomplish this, we consider the following conditions, corresponding closely to those outlined in Rosenbaum and Rubin (1983).

'Treatment assignment' of individual $k \in U$, here interpreted as the individual's possible inclusion in the Web panel, is indicated by the variable z_k:

$$z_k = \begin{cases} 1 & \text{if } k \in U_W \\ 0 & \text{if } k \notin U_W \end{cases} \tag{2}$$

The treatment assignment is assumed to be *strongly ignorable* given a random vector $\boldsymbol{x}_k$ of covariates; that is, y_k $(k \in U)$ is conditionally independent of z_k given $\boldsymbol{x}_k$. It follows that the conditional expected value of y_k given $\boldsymbol{x}_k$, $E(y_k \,|\boldsymbol{x}_k)$, is independent of z_k. If treatment assignment is strongly ignorable given $\boldsymbol{x}_k$, it is strongly ignorable given any function of $\boldsymbol{x}_k$–any *balancing score*–such that $\boldsymbol{x}_k$ is conditionally independent of z_k given $b(\boldsymbol{x}_k)$. One implication of this is that the conditional expected value of y_k given $b(\boldsymbol{x}_k)$, $E(y_k \,|b(\boldsymbol{x}_k))$, is independent of z_k. The coarsest balancing score is the *propensity score*, $e\,(\boldsymbol{x}_k)$, defined as

$$e\,(\boldsymbol{x}_k) = \Pr\,(z_k = 1 \,|\boldsymbol{x}_k\,) \tag{3}$$

(the finest balancing score is $\boldsymbol{x}_k$ itself.) Now assume that the propensity scores of all individuals in U are known. Theoretically, the propensity scores may assume any values between 0 and 1. This limits their practical use somewhat, since the number of individuals with the same propensity score may be equal or close to zero. It seems plausible, however, that individuals with similar propensity scores have similar conditional expected values. Thus, we assume that if U is divided into a large number H of classes, $U_1, ..., U_h, ..., U_H$, with each class containing individuals with similar propensity scores, then the individuals within class share a common conditional mean and variance. Formally, we assume that

$$E(y_k \,|e(\boldsymbol{x}_k)\,) = \mu_h; \qquad V(y_k \,|e(\boldsymbol{x}_k)\,) = \sigma_h^2 \tag{4}$$

for all $k \in U_h$ $(h = 1, ..., H)$. Then, μ can be written as

$$\mu = \sum_{h=1}^{H} D_h \mu_h \tag{5}$$

where D_h denotes the probability that an individual, randomly selected from U, belongs to class U_h $(h = 1, ..., H)$.

Estimation of μ requires knowledge of the class membership of each individual $k \in s$. For $h = 1, ..., H$, let the intersection $s_W \cap U_h$ be denoted s_{Wh} (of random size n_{Wh}), the intersection $s_T \cap U_h$ be denoted s_{Th} (of random size n_{Th}), and let the union $s_{Wh} \cup s_{Th}$ be denoted s_h(of random size n_h). Assuming class membership to be known for sampled individuals, we propose the following sample-based estimates of D_h and μ_h. First, since s_T is chosen by SI, the distribution of s_T over classes is likely to resemble the corresponding population distribution over classes (if n_T is sufficiently large.) Thus, it makes sense to estimate D_h by $d_h = n_{Th}/n_T$. Second, since treatment assignment is strongly ignorable, estimation of the class means μ_h can be based solely on s_{Wh}. This motivates the estimation of μ_h by the class mean of the Web sample: $\bar{y}_{s_{Wh}} = \sum_{s_{Wh}} y_k/n_{Wh}$. The resulting estimator of μ is:

$$\bar{y}_s = \sum_{h=1}^{H} d_h \bar{y}_{s_{Wh}}. \tag{6}$$

In practice, the propensity scores must be estimated for $k \in s$, which calls for some additional modeling. A strategy that lies near at hand is to formulate a logistic regression model for $e\,(\boldsymbol{x}_k)$ as function of $\boldsymbol{x}_k$, and estimate the propensity scores under this model. Then, the sample s is divided into classes with similar estimated propensity scores. Hopefully, the sample classes coincide reasonably well with the classes in the population.

The estimator $\bar{y}_s$, being intuitively appealing, is already in use in various Web surveys (see, for instance, Terhanian et al, 2001). In this paper, we use model assumptions to derive its expectation and variance and, most importantly, suggest an estimator of its variance.

2 The propensity score weighting procedure

The propensity score weighting procedure of interest in this paper includes the following steps:

1. Estimation of $e(\boldsymbol{x}_k)$ for each $k \in s$.
2. Division of the sample s into classes containing individuals with similar estimated $e(\boldsymbol{x}_k)$:s.
3. Estimation of μ.

In this section, we discuss some features of steps 1 and 2.

In step 1, the propensity scores $e(\boldsymbol{x}_k)$ are estimated by use of the indicator variable z_k and the vector $\boldsymbol{x}_k$ of covariates, both available for all $k \in s$. The covariates (sometimes referred to as "webographics") might concern lifestyle, attitudes and self-perception. A standard logistic regression model for $e(\boldsymbol{x}_k)$ as function of $\boldsymbol{x}_k$ is formulated (Neter et al (1996, eqn. (14.37)), Manly (1994, eqn. (8.3))), according to which the z_k:s ($k \in U$) are independent Bernoulli random variables with conditional expected values:

$$E(z_k \,|\boldsymbol{x}_k) = e(\boldsymbol{x}_k) = \frac{\exp\left(\boldsymbol{\beta}'\boldsymbol{x}_k\right)}{1+\exp\left(\boldsymbol{\beta}'\boldsymbol{x}_k\right)} \qquad (7)$$

where

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{bmatrix}; \quad \boldsymbol{x}_k = \begin{bmatrix} 1 \\ x_{1k} \\ \vdots \\ x_{p-1,k} \end{bmatrix}$$

If s is an SI sample from U, then (as in type (1) in Manly, 1994, p. 120), application of logistic regression is straight-forward, and $e(\boldsymbol{x}_k)$ is estimated by

$$\hat{e}(\boldsymbol{x}_k) = \frac{\exp\left(\mathbf{b}'\boldsymbol{x}_k\right)}{1+\exp\left(\mathbf{b}'\boldsymbol{x}_k\right)} \qquad (8)$$

where $\mathbf{b}$ is a vector of maximum likelihood (ML) estimates of $\beta_0, \beta_1, ..., \beta_{p-1}$. In our case, s_W and s_T are lumped together to form s. As shown in Seber (1984, p. 312) (and discussed in Manly, 1994, sec. 8.10), for lumped data, the model in eqn. (7) needs modification. In our setting, the intercept β_0 should be reduced by

$$\log_e\left[\frac{n_W(1-P_W)}{n_T P_W}\right], \qquad (9)$$

where $P_W = N_W/N$ is the Web panel fraction of the total population. (The ML-estimate of β_0 must of course be adjusted correspondingly.)

Next, the total sample s is divided into weighting classes containing individuals with similar estimated propensity scores. In the literature, you sometimes see the recommendation to form a few (around five) groups, and to make them of equal size in terms of n_{Th}. The recommendation is based on an early paper by Cochran (1968), in which subclassification by a single covariate is considered. In our setting, the division of the sample should aim at forming groups with similar propensity scores. Then, it does not make sense to create groups of equal size. Instead, the group members' closeness in $\hat{e}(\boldsymbol{x})$ is crucial.

3 Statistical modeling

In order to derive the statistical properties of $\bar{y}_s$, we use statistical models for y_k and the vector $\boldsymbol{n}_T = (n_{T1}, ..., n_{Th}, ..., n_{TH})$. In this section, our models are formulated, and the corresponding statistical properties of $\bar{y}_s$ investigated. Please note, however, that several potential sources of bias and variance are ignored, including

- the choice of x variables included in the logistic regression model,
- the fit of the logistic regression model, and
- the division of the Web sample into classes by $\hat{e}(\boldsymbol{x})$ instead of $e(\boldsymbol{x})$.

As will soon be discussed, we also ignore the randomness of $n_{W1}, ..., n_{WH}$.

Our approach relies on the following random models for y_k and $\boldsymbol{n}_T$.

Model $\mathbf{m_1}$
Conditional on $e(\boldsymbol{x})$, the study variable values y_k for $k \in s_h$, $h = 1, ..., H$, are iid random variables with expectation $E_{m_1}(y_k) = \mu_h$ and variance $V_{m_1}(y_k) = \sigma_h^2$.

From model m_1 (and general properties of a random sample), the conditional expectation and variance of $\bar{y}_{s_{Wh}}$ (conditional on $e(\boldsymbol{x})$ and n_{Wh}) is $E_{m_1}(\bar{y}_{s_{Wh}}) = \mu_h$ and $V_{m_1}(\bar{y}_{s_{Wh}}) = \sigma_h^2/n_{Wh}$, respectively. Also, $\bar{y}_{s_{Wh}}$ and $\bar{y}_{s_{Wi}}$ $(h, i = 1, ..., H; i \neq h)$ are independent. Since s_{Wh} is not a probability sample, the statistical properties of n_{Wh} are unknown. Therefore, throughout our analysis, we condition on n_{Wh}.

Model $\mathbf{m_2}$
Each individual $k \in s_T$ is independently assigned membership to one out of H classes. For each individual, the probability of being assigned to class h is D_h. Thus, the random vector $\boldsymbol{n}_T$ has a *multinomial distribution* with n_T trials, H possible outcomes, and cell probabilities $D_1, ..., D_H$.

Under model m_2, the marginal distribution of n_{Th} $(h = 1, ..., H)$ is binomially distributed with parameters n_T and D_h. It follows that the expectation and variance of n_{Th} is $E_{m_2}(n_{Th}) = n_T D_h$ and $V_{m_2}(n_{Th}) = n_T D_h (1 - D_h)$, respectively.

In addition to model m_1 and m_2, we assume that $\bar{y}_{s_{Wh}}$ and d_h $(h = 1, ..., H)$ are independent. Hopefully, the dependency between the statistics is not very large, and our assumption is not an oversimplification.

4 Statistical properties of $\bar{y}_s$

The expectation and approximate variance of $\bar{y}_s$, based on the models formulated in sec. 3, are given in Theorem 4.1. The theorem is proved in appendix.

Theorem 4.1 *Under model m_1 and m_2, the estimator $\bar{y}_s$ is model-unbiased for μ. The variance of $\bar{y}_s$ is given by*

$$V_{m_1 m_2}(\bar{y}_s) = V_1 + V_2 \tag{10}$$

where

$$V_1 = \frac{1}{n_T} \sum_{h=1}^{H} \left[D_h (\mu_h - \mu)^2 + D_h (1 - D_h) \frac{\sigma_h^2}{n_{Wh}} \right]$$

and

$$V_2 = \sum_{h=1}^{H} D_h^2 \frac{\sigma_h^2}{n_{Wh}}.$$

We construct an estimator of $V_{m_1 m_2}(\bar{y}_s)$ by use of the "method of moments" (Casella and Berger, 1990, ch. 7). In this context, it means that we replace the unknown model parameters in the variance expression by their sample analogues. This gives the estimator

$$\hat{V}(\bar{y}_s) = \hat{V}_1 + \hat{V}_2 \tag{11}$$

where

$$\hat{V}_1 = \frac{1}{n_T} \sum_{h=1}^{H} \left[d_h (\bar{y}_{s_{Wh}} - \bar{y}_s)^2 + d_h (1 - d_h) \frac{s_{Wh}^2}{n_{Wh}} \right],$$

$s_{Wh}^2 = \sum_{k \in s_{Wh}} (y_k - \bar{y}_{s_{Wh}})^2 / (n_{Wh} - 1)$, and

$$\hat{V}_2 = \sum_{h=1}^{H} d_h^2 \frac{s_{Wh}^2}{n_{Wh}}.$$

The method of moments is intuitively rather than theoretically motivated. In consequence, we expect $\hat{V}(\bar{y}_s)$ to be biased for the true variance. The size of the bias is investigated in a simulation, see sec. 5.

5 Simulation

In this section, we familiarize ourselves with $\bar{y}_s$ and $\hat{V}(\bar{y}_s)$ through a simulation. We create an artificial target population, draw a large number of independent samples from the same, and use these samples to investigate the estimators' statistical properties.

5.1 Creation of the target population

An artificial target population U of elements of size $N = 50,000$ is constructed as follows.

Covariates We simulate N values of a bivariate standard normal distribution

$$(X_1, X_2) \sim N(\mathbf{0}, \boldsymbol{\Sigma})$$

with covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Class index h	Conditions		
	x_1	x_2	x_3
1	≤ 0	≤ 0	0
2	≤ 0	> 0	0
3	> 0	≤ 0	0
4	> 0	> 0	0
5	≤ 0	≤ 0	1
6	≤ 0	> 0	1
7	> 0	≤ 0	1
8	> 0	> 0	1

Table 1: Division into classes.

and N values of the Bernoulli distributed variable

$$X_3 \sim Be\left(\frac{\exp(\gamma_0 + \gamma_1 X_1 + \gamma_2 X_2)}{1 + \exp(\gamma_0 + \gamma_1 X_1 + \gamma_2 X_2)}\right).$$

This produces two continuous and one discrete covariate. The model parameters are set to $\rho = .5$, $\gamma_0 = 0$, and $\gamma_1 = \gamma_2 = 1$.

Division into classes We use the realized values on the covariates to partition U into $H = 8$ classes, $U_1, ..., U_h, ..., U_H$, in accordance with Table 1. The realized size of a class U_h is denoted N_h.

Study variable For class U_h $(h = 1, ..., 8)$, we simulate N_h values of a study variable as

$$Y_h \sim N\left(\mu_h, \sigma_h^2\right)$$

where $\mu_1 = -0.4$ and $\mu_h = \mu_{h-1} + 0.1$ for $h > 1$, and $\sigma_h = \sqrt{\sigma_h^2} = \lambda_0 + \lambda_1 |\mu_h|$. The model parameters are set to $\lambda_0 = \lambda_1 = 1$. Note that in this way, we get different study variable means for different classes, larger means for larger values on the covariates, and variances proportional to the level of the means.

Treatment assignment For class U_h $(h = 1, ..., 8)$, we simulate N_h values of the Bernoulli variable

$$Z_h \sim Be(\theta_h)$$

where $\theta_1 = 0.1$ and $\theta_h = \theta_{h-1} + 0.1$ for $h > 1$. In this way, the treatment assignment is dependent on all auxiliary variables (through the forming of the classes). Furthermore, treatment assignment is strongly ignorable in the sense discussed in sec. 1.2.

5.2 Sampling from the artificial population

From the population, $R = 10,000$ independent samples $s_{(1)}, ..., s_{(r)}, ..., s_{(R)}$ are drawn. Each sample $s_{(r)}$ is the union of $s_{T(r)}$ and $s_{W(r)}$, where $s_{T(r)}$ is an SI sample from U and $s_{W(r)}$ an SI sample from U_W. Throughout, the sizes of $s_{T(r)}$ and $s_{W(r)}$ are $n_{T(r)} = 1000$ and $n_{W(r)} = 5000$, respectively.

5.3 Estimation and results

In the estimation, the class membership of each sampled individual is taken to be known. Thus, we limit our attention to the favorable case in which there is no uncertainty in the division of $s_{(r)}$ into classes. For $r = 1, ..., R$, we calculate a propensity score estimate $\bar{y}_{s(r)}$ in accordance with eqn. (6). In addition, we calculate the variance estimates $\hat{V}_{1(r)}$ and $\hat{V}\left(\bar{y}_{s(r)}\right)$ in accordance with the formulae for $\hat{V}_1$ and $\hat{V}(\bar{y}_s)$, respectively, in eqn. (11). Averages of the sample estimates are calculated as

$$\overline{\bar{y}}_{s(r)} = \frac{1}{R}\sum_{r=1}^{R} \bar{y}_{s(r)}; \qquad \overline{\hat{V}}_1 = \frac{1}{R}\sum_{r=1}^{R} \hat{V}_{1(r)};$$

$$\overline{\hat{V}}\left(\bar{y}_{s(r)}\right) = \frac{1}{R}\sum_{r=1}^{R} \hat{V}\left(\bar{y}_{s(r)}\right) \tag{12}$$

and an approximation of the true variance of $\bar{y}_s$ as

$$V(\bar{y}_s) = \frac{1}{N}\sum_{r=1}^{R}\left(\bar{y}_{s(r)} - \overline{\bar{y}}_{s(r)}\right)^2. \tag{13}$$

In fig. 1, the frequency distribution of the estimated relative bias in $\bar{y}_s$, $\left(\bar{y}_{s(r)} - \mu\right)/\mu$, is shown. On the average, the relative bias is very close to zero:

$$\frac{\overline{\bar{y}}_{s(r)} - \mu}{\mu} = -.005.$$

This result was expected, since the artificial population is constructed in agreement with our model assumptions.

Fig. 2 shows the frequency distribution of the ratio $\hat{V}_{1(r)}/\hat{V}\left(\bar{y}_{s(r)}\right)$. On the average,

$$\frac{\overline{\hat{V}}_1}{\overline{\hat{V}}\left(\bar{y}_{s(r)}\right)} = .642,$$

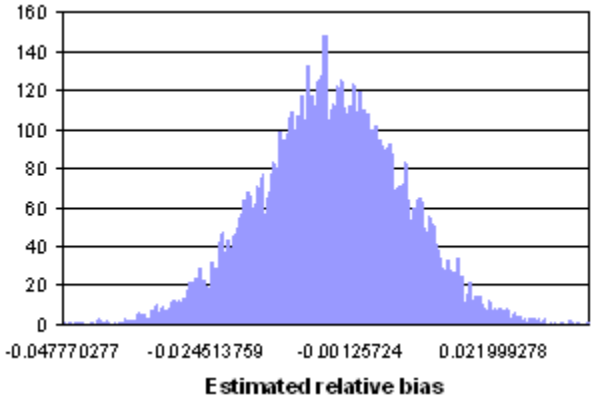

Figure 1: Frequency distribution of $\left(\bar{y}_{s(r)} - \mu\right)/\mu$.

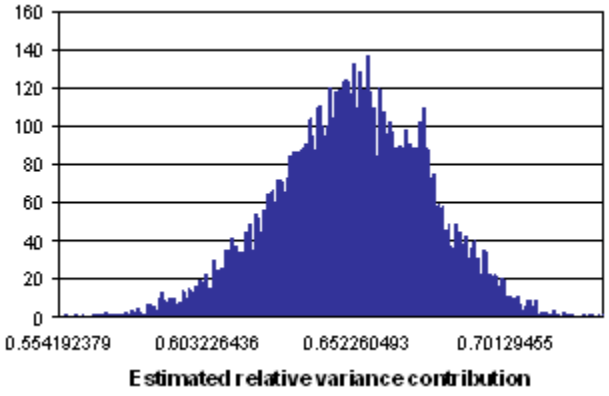

Figure 2: Frequency distribution of $\hat{V}_{1(r)}/\hat{V}\left(\bar{y}_{s(r)}\right)$.

which illustrates that the term V_1 may represent a large proportion of the total variance.

The relative bias of $\overline{\hat{V}}\left(\bar{y}_{s(r)}\right)$, finally, is approximately given by

$$\frac{\overline{\hat{V}}\left(\bar{y}_{s(r)}\right) - V\left(\bar{y}_s\right)}{V\left(\bar{y}_s\right)} = .113,$$

revealing that the suggested variance estimator is quite conservative.

6 Conclusions and final remarks

The propensity score estimator has developed from statistical practice and its needs, rather than as a theoretical exercise. This is probably the reason why its theoretical motivation is not entirely clear from the literature. In this paper, we have formulated a simple (ideal) model world, in which the propensity score estimator of the population mean is unbiased for the same. In this setting, it is a straight-forward task to develop an expression for the estimator's variance. By replacing unknown entities in the variance formula by their sample counterparts, we arrive at an intuitive variance estimator.

Our variance expression consists of two terms, V_1 and V_2, where the second term resembles the variance of a poststratified estimator. One might feel tempted to confine oneself to estimating V_2. It is however easy to conceive of situations when this would lead to serious underestimation of the total variance; for instance, if the class means differ a lot, or if the telephone sample is small.

In a simulation study, we have made sure that the propensity score estimator really is unbiased for the true mean if the model assumptions hold. We have demonstrated that the variance term V_1 may represent a large proportion of the total variance, and discovered that our variance estimator is likely to overestimate the true variance.

In the simulation, the propensity score of each sampled individual was known. In reality, they must however be estimated from the sample data. Further simulations are necessary to investigate the impact of this additional step on the propensity score estimator. The consequences of deviating

from the strong ignorability assumption also remain to be investigated.

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A Proof of Theorem 3.1

We start with the expectation. Using the conditional independency of $\bar{y}_{s_{Wh}}$ and d_h,

$$E_{m_1m_2}(\bar{y}_s) = \sum_{h=1}^{H} E_{m_2}(d_h)\, E_{m_1}(\bar{y}_{s_{Wh}}) = \sum_{h=1}^{H} D_h\mu_h.$$

Now let us turn to the variance. By general properties of the variance of a sum of random variables (see Ross (1997, eqn. (2.16)),

$$\begin{aligned} V_{m_1m_2}(\bar{y}_s) &= \sum_{h=1}^{H_s} V_{m_1m_2}(d_h\bar{y}_{s_{Wh}}) \\ &\quad +2\sum_{h=1}^{H_s}\sum_{i<h} Cov_{m_1m_2}(d_h\bar{y}_{s_{Wh}}, d_i\bar{y}_{s_{Wi}}) \\ &= V_1 + V_2 \end{aligned}$$

where $Cov_{m_1m_2}(d_h\bar{y}_{s_{Wh}}, d_i\bar{y}_{s_{Wi}})$ is the covariance between $d_h\bar{y}_{s_{Wh}}$ and $d_i\bar{y}_{s_{Wi}}$.

Consider any term $V_{m_1m_2}(d_h\bar{y}_{s_{Wh}})$ in V_1. Since d_h and $\bar{y}_{s_{Wh}}$ are independent,

$$\begin{aligned} V_{m_1m_2}(d_h\bar{y}_{s_{Wh}}) &= [E_{m_1}(\bar{y}_{s_{Wh}})]^2 V_{m_2}(d_h) \\ &\quad + [E_{m_2}(d_h)]^2 V_{m_1}(\bar{y}_{s_{Wh}}) \\ &\quad + V_{m_2}(d_h) V_{m_1}(\bar{y}_{s_{Wh}}) \\ &= \mu_h^2 \frac{D_h(1-D_h)}{n_T} + D_h^2\frac{\sigma_h^2}{n_{Wh}} \\ &\quad + \frac{D_h(1-D_h)}{n_T}\frac{\sigma_h^2}{n_{Wh}}, \end{aligned}$$

and

$$\begin{aligned} V_1 &= \sum_{h=1}^{H}\left[\mu_h^2\frac{D_h(1-D_h)}{n_T} + D_h^2\frac{\sigma_h^2}{n_{Wh}} \right. \\ &\quad \left. + \frac{D_h(1-D_h)}{n_T}\frac{\sigma_h^2}{n_{Wh}}\right]. \end{aligned}$$

Now consider any covariance term in V_2:

$$\begin{aligned} & Cov_{m_1m_2}(d_h\bar{y}_{s_{Wh}}, d_i\bar{y}_{s_{Wi}}) \\ = \; & E_{m_1m_2}(d_h\bar{y}_{s_{Wh}} d_i\bar{y}_{s_{Wi}}) \\ & - E_{m_1m_2}(d_h\bar{y}_{s_{Wh}})\, E_{m_1m_2}(d_i\bar{y}_{s_{Wi}}) \\ = \; & E_{m_1m_2}(d_h\bar{y}_{s_{Wh}} d_i\bar{y}_{s_{Wi}}) \\ & - D_h\mu_h D_i\mu_i. \end{aligned}$$

By use of conditioning,

$$\begin{aligned}
& E_{m_1m_2}(d_h\bar{y}_{s_{Wh}}d_i\bar{y}_{s_{Wi}}) \\
= \; & E_{m_1}[\bar{y}_{s_{Wh}}\bar{y}_{s_{Wi}}E_{m_2}(d_hd_i\,|m_1\,)] \\
= \; & E_{m_1}\Bigg\{\bar{y}_{s_{Wh}}\bar{y}_{s_{Wi}}\left[\frac{1}{n_T^2}Cov_{m_2}(n_{Th}n_{Ti}\,|m_1\,)\right. \\
& \left. +\; E_{m_2}(d_h\,|m_1\,)\,E_{m_2}(d_i\,|m_1\,)\right]\Bigg\} \\
= \; & E_{m_1}\Bigg\{\bar{y}_{s_{Wh}}\bar{y}_{s_{Wi}}\left[\frac{1}{n_T^2}Cov_{m_2}(n_{Th}n_{Ti}\,|m_1\,)\right. \\
& \left. +\; D_hD_i\right]\Bigg\}
\end{aligned}$$

where $Cov_{m_2}(n_{Th}n_{Ti}\,|m_1\,)$ is the covariance between n_{Th} and n_{Ti}. Since n_{Th} and n_{Ti} belong to a multinomial distribution, from Agresti (1990, p. 44),

$$Cov_{m_2}(n_{Th}n_{Ti}\,|m_1\,) = -n_TD_hD_i,$$

and we arrive at

$$\begin{aligned}
& E_{m_1m_2}(d_h\bar{y}_{s_{Wh}}d_i\bar{y}_{s_{Wi}}) \\
= \; & E_{m_1}\left\{\bar{y}_{s_{Wh}}\bar{y}_{s_{Wi}}\left[D_hD_i\left(1-\frac{1}{n_T}\right)\right]\right\} \\
= \; & D_hD_i\left(1-\frac{1}{n_T}\right)E_{m_1}(\bar{y}_{s_{Wh}}\bar{y}_{s_{Wi}}) \\
= \; & D_hD_i\left(1-\frac{1}{n_T}\right)\mu_h\mu_i.
\end{aligned}$$

Thus, V_2 is given by

$$\begin{aligned}
V_2 \; = \; & 2\sum_{h=1}^{H}\sum_{i<h}\left[D_hD_i\left(1-\frac{1}{n_T}\right)\mu_h\mu_i\right. \\
& \left. -\; D_h\mu_hD_i\mu_i\right] \\
= \; & -\frac{2}{n_T}\sum_{h=1}^{H}\sum_{i<h}D_hD_i\mu_h\mu_i \\
= \; & \frac{1}{n_T}\left[\sum_{h=1}^{H}\mu_h^2D_h^2-\left(\sum_{h=1}^{H}\mu_hD_h\right)^2\right].
\end{aligned}$$

Finally, we add V_1 and V_2:

$$\begin{aligned}
V_1+V_2 \; = \; & \sum_{h=1}^{H}\left[\mu_h^2\frac{D_h\,(1-D_h)}{n_T}+D_h^2\frac{\sigma_h^2}{n_{Wh}}\right. \\
& \left. +\;\frac{D_h\,(1-D_h)}{n_T}\frac{\sigma_h^2}{n_{Wh}}\right] \\
& +\frac{1}{n_T}\left[\sum_{h=1}^{H}\mu_h^2D_h^2-\left(\sum_{h=1}^{H}\mu_hD_h\right)^2\right] \\
= \; & \frac{1}{n_T}\left[\sum_{h=1}^{H}\mu_h^2D_h-\sum_{h=1}^{H}\mu_h^2D_h^2\right. \\
& +D_h\,(1-D_h)\,\frac{\sigma_h^2}{n_{Wh}}+\sum_{h=1}^{H}\mu_h^2D_h^2 \\
& \left. -\left(\sum_{h=1}^{H}\mu_hD_h\right)^2\right]+\sum_{h=1}^{H}D_h^2\frac{\sigma_h^2}{n_{Wh}}
\end{aligned}$$

$$\begin{aligned}
= \; & \frac{1}{n_T}\left\{\left[\sum_{h=1}^{H}\mu_h^2D_h-\left(\sum_{h=1}^{H}\mu_hD_h\right)^2\right]\right. \\
& \left. +\;D_h\,(1-D_h)\,\frac{\sigma_h^2}{n_{Wh}}\right\} \\
& +\sum_{h=1}^{H}D_h^2\frac{\sigma_h^2}{n_{Wh}} \\
= \; & \frac{1}{n_T}\sum_{h=1}^{H}D_h\left[\left(\mu_h-\bar{\mu}_{(y|e\,)}\right)^2+(1-D_h)\,\frac{\sigma_h^2}{n_{Wh}}\right] \\
& +\sum_{h=1}^{H}D_h^2\frac{\sigma_h^2}{n_{Wh}}
\end{aligned}$$

which equals the stated expression for $V_{m_1m_2}(\bar{y}_s)$.