SAMPLE DESIGN AND ESTIMATION OF LIFESPANS OF FEDERAL RESERVE NOTES

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Abstract: The Federal Reserve (FR) maintains several time series pertaining to FR notes, including number paid out, number in circulation, and the numbers inspected and destroyed at each of the 37 FR regional offices. In a steady state, the average lifespan of notes is a rather straightforward calculation based on these series. However, recently the series have been unstable, and a survey of notes about to be destroyed at the FR offices was conducted to provide a direct estimate of the average lifespan of notes. Outside data on the ages of a cross-section of notes were used to estimate the variation of lifespans and a required sample size for this survey, and the destruction rates were used to estimate the number of offices (clusters) that should be included. The serial number of each sample note was recorded so that we could match with existing data to determine when the note was first shipped to the FR office, yielding the note's gross lifespan. Several methods were used to impute shipment dates in cases where the match failed. Finally, inventory models were used to estimate the notes' shelf lives. We concluded that the average lifespan of notes has increased about twenty percent over the last decade.

OBJECTIVES OF THE STUDY

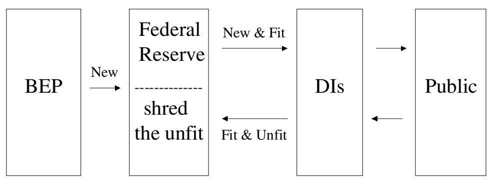
The goal of this study is to estimate the average lifespan of Federal Reserve (FR) $1 notes. Lifespan is defined to be the length of time a note spends in circulation before being declared unfit and destroyed at a FR office. The one-dollar denomination was selected for our study because it is the highest-volume note in circulation and represents the largest volume in the FR's annual order for banknotes (about 50% of the annual print order for FY2003).

BACKGROUND

In order to supply the demands of the public for currency, the FR purchases notes from the Bureau of Engraving and Printing (BEP). To the extent that the lifespan of notes can be extended, the purchases of new notes can be reduced, with corresponding reductions in FR expenditures. In the mid-1990s, in an attempt to increase the durability of currency paper, the BEP requested that its paper manufacturer, Crane and Company, consider changes to its processes that would improve the life of the paper substrate. In response, Crane developed several paper-optimization techniques that appeared to improve banknote durability. For example, the use of bleach, which acts to break down fiber, has been eliminated from the process of making the paper. The outcome of Crane's efforts was not known until the late 1990s, when the Federal Reserve's analysis of currency data suggested a downward trend in destruction rates, and a corresponding increase in the life of $1 notes. The present study attempts to measure the extent of this increase.

Figure 1 presents a schematic of the currency circulation process. New currency is supplied to the FR by the BEP. The FR supplies depository institutions (DIs) with currency taken from its inventories of both new and fit dollars. The DIs supply currency to the public and absorb the public's excess holdings, then return suspected unfit currency and any seasonal excess to the FR offices. The FR offices inspect these notes and determine which notes are fit for further circulation and destroy (shred) the unfit notes. So orders of new notes from BEP are required to accommodate any increases in the public's demand for notes and to replace notes that were shredded by the FR offices.

Figure 1
Model of the Currency Circulation Process



We have available for our study several monthly time series pertaining to FR notes, including number of new notes received from BEP, number of new notes paid into circulation (NPO), number of notes in circulation (i.e., the stock of notes demanded by the public), and the numbers inspected and destroyed at each of the 37 FR offices.

Over the years, the FR has estimated average note lifespans using inventory models based on the just-mentioned time series. In the steady state, one can write:

$$C = NPO \times \overline{L}$$

where C is the number of notes in circulation, NPO is the number of new notes paid out by the FR each month, and $\overline{L}$ is the average lifespan of notes. In a study conducted by FR in the late 1980s, the average lifespan of one-dollar notes was estimated in this fashion to be 17.5 months.

The inventory equation works well when the series in question are stable. But, to the extent that there is variability in C and/or NPO, one must decide just exactly what to use for C and NPO in the equation (e.g., should one average over how many and which months). In recent years the series have been so unstable that our confidence in the inventory model was shaken, and the FR decided to attempt to estimate average lifespan more directly -- in particular, to measure the ages of a sample of notes whose lives are about to end (i.e., about to be shredded by a FR office) and then average the implied lifespans.

SAMPLING DESIGN

Preliminary Estimate of Sample Size

In order to determine the number of notes needed for the study, we needed to have some knowledge about the variance of lifespans. Unfortunately, we don't have any historical data on the lifespans of individual notes. But we do have a data set of ages of a sample of a cross-section of notes in circulation in 1989 (described in "A Comprehensive Assessment of U.S. Currency Quality, Age, & Cost Relationships" which we will refer to in this paper as the "Comprehensive Study.")

In the Comprehensive Study, a sample of notes was supplied by DIs from the population of notes that they had received from the public during the sample period (see the lower rightmost arrow in Figure 1). The FR measured the age of each note, defined to be the time between when the note was first sent to a DI by a FR office and the date it appeared in the sample. Figure 2 shows the estimated age distribution of notes in circulation, converted to a steady state by removing the seasonal and cyclical influences of the number of new notes issued by the FR (i.e., the monthly NPO series).

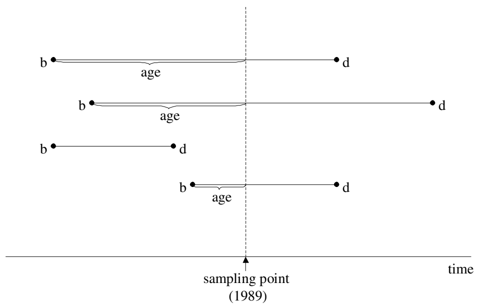
Figure 2
Estimated Distribution of Ages of $1 Notes in Circulation,1989

If a note was randomly selected from the circulation, we can regard the sampling point as uniformly distributed within the note's lifespan (Figure 3). In other words, the age of a note randomly selected from circulation can be regarded as the product of a Uniform random variable and the lifespan of the note:

$$A = U \times L$$

where A is the age of a note sampled from circulation, U has the uniform distribution U(0,1), L represents the lifespan of the note, and U and L are independent.

Figure 3
Illustration of relationship between age and lifespan of a note



b: "birth," the date a note was first paid into circulation by a FR office
d: "destroyed," the date the note was shredded at a FR office.

From the above equation relating age to lifespan of notes randomly selected from circulating notes, we can derive the expectation and variance of L, the lifespans of notes that existed as of the sampling point, relative to those of A:

$$\mu_L = 2\mu_A$$
$$\sigma_L^2 = 3\sigma_A^2 - \mu_A^2$$

Note that the sampling of ages and lifespans in the Comprehensive Study was *length-biased* in the sense that longer-lived notes had a greater probability of being intercepted than shorter-lived notes. These equations refer to means and variances of intercepted notes in circulation, not of all notes that have been introduced into circulation. The mean of this latter whole distribution would be smaller than the mean of the length-biased distribution. The inequality is reversed for the variances. We have been unable to derive a general expression for the variance of the whole distribution of L in terms of the parameters of the intercepted A distribution, but it appears that the formula may be understating the desired variance by as much as one half. For example, if the whole distribution of L is exponential, the variance of the whole distribution of L would be 12/5 times the result given by our formula. (Experiments with other assumed distributions gave less extreme differences.)

The Comprehensive Study estimates of the parameters of the age distribution were

$$\hat{\mu}_A = 12.1 \text{ months}$$
$$\hat{\sigma}_A = 12.3 \text{ months}$$

Our formula for the standard deviation of lifespans ($\hat{\sigma}_L$) yields 17.5 months, which we adjust upward to 24.7 months to adjust for the potential length-bias. Our client desired a 99% confidence interval of e=±1 month. Assuming the normal distribution for the sample average of lifespans, we estimated the required size of a simple random sample to be:

$$n = z^2 \hat{\sigma}_L^2 / e^2 = (2.58)^2 (24.7)^2 / 1^2 = 4061$$

Cluster sampling

It would not be feasible to draw a simple random sample from the notes at the points of destruction. Instead, the plan was to select a sample of FR offices, which in turn would be asked to select random samples of their unfit notes. Thus, we have a two-stage cluster design with the offices being the clusters.

Using existing historical data on the destruction rates at each office, and assuming a rough proportionality between the variation of destruction rates and the variation of average lifespans across office, we concluded that we needed about half of the FR offices. Our client decided it would be more practicable to just ask all of the offices (clusters) to participate.

Acknowledging that the FR offices might not get a true random sample within their populations, we recommended that our client attempt to compensate by increasing the sample to average at least 200 per office and be roughly proportional to office volume, that is, the number of notes destroyed by the office in 2001. Our client rounded up the number of notes that each office would collect, which increased the sample size from 7400 to nearly 9000 notes.

Sampling time frame

When choosing the length of the study period, we attempted to avoid the survey bias that could be caused by some special depositing activities. For example, some depository institutions represent a significant share of daily notes deposited with a FR Bank office. If the survey was conducted for only one day, the fitness of the notes processed could be determined by only a few depository institutions and would not reflect the general condition of notes that were deposited. To guard against possible day-to-day variation, the entire week of July 29, 2002 was chosen for our data collection.

Data Collection

The FR offices collected the assigned number of one-dollar notes that the fitness sensors had marked for destruction. Machine operators removed these marked-for-destruction notes and recorded specific information for the survey, including the series year, serial number, processing machine, and machine shift. Based on a note's serial number, the FR Board's Currency Ordering System (COS) tracked when the BEP had first shipped these notes to each Reserve Bank, and then we calculated the approximate gross lifespan of the notes at the time of destruction.

ESTIMATION OF LIFESPANS

Missing Data

There were about 400 sampled notes with serial numbers which we were unable to match with shipment information in the COS database. Among these notes, many were not in the COS database because COS does not have information on series before 1990. We explored a few imputation methods for these no-shipment-record notes and decided rather arbitrarily to assign the midpoint of period between the year of series in question and the following series. For example, if 1981 and 1985 were successive series, then a Series 1981 note was given a BEP shipment date of January 1, 1983. Using this method, we were able to estimate shipment date for about 200 notes. A small number of sample notes (about 200) were excluded from our analysis because of lack of shipment date information.

Shelf life

While we have BEP shipment date by serial number, we do not have the date the note was initially sent out to a DI from a FR office. As noted earlier, in this study, a note is defined to be "born" when it is first paid out by a FR office. So a note's birth date is determined by adding to its BEP shipment date the amount of time it stayed in a FR office's vault prior to its initial payout to a DI. This time period, or shelf life, was estimated as follows: Employing the FR Board's monthly data of inventories, receipts, and payouts of new currency for each FR office, we computed the average shelf life for new currency using an inventory model based on the FR System's first in/first out currency inventory policy:

$$I = NPO \times \bar{S}$$

where I is the inventory of notes at the FR system, NPO is the number of new notes paid out by the FR each month, and $\bar{S}$ is the average shelf life of notes.

New currency payout behavior can change over time due to demand, availability, and changes in inventory of fit currency; thus we adjusted the inventory model to estimate the shelf life for a given month by incorporating 12-month moving averages of inventory and NPO.

RESULTS

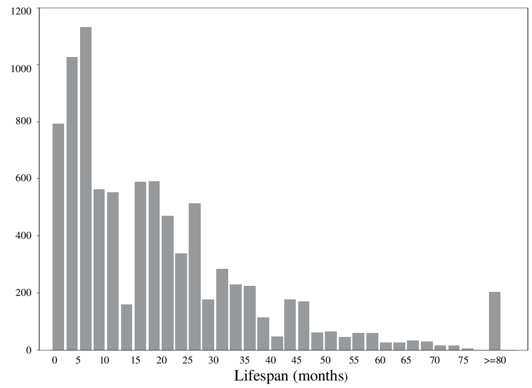
If we call the difference between the BEP shipment date and sampling point the gross lifespan, then the lifespan for each note would be estimated by subtracting an estimate of the shelf life from its gross lifespan:

$$L_i = G_i - S_i.$$

Using these estimated lifespans of the individual sampled notes, we estimated that the average life of the one-dollar notes in our study was about 20.6 months with a standard error of 0.3 months.

Figure 4 illustrates the sample frequencies of estimated lifespans of one-dollar notes. The survey data are represented by a positively skewed curve, with the greatest frequency of destroyed notes occurring at young ages. On the other hand, some notes lasted quite a long time – more than 80 months in some cases (grouped into one bar in Figure 4).

Figure 4
Sample Frequencies of Estimated Lifespans of $1 Notes



CONCLUSION

Based on our cluster sample of about to be destroyed notes, we concluded that the average lifespan of $1 notes has increased by about 20 percent over the past decade – from 17.5 to 21 months. So we would expect a twenty percent reduction in FR expenditures on new notes.

REFERENCE

"A Comprehensive Assessment of U.S. Currency Quality, Age, & Cost Relationships," Federal Reserve System, September 1991