COMPARISON OF VARIANCE ESTIMATION TECHNIQUES FOR A PRICE INDEX WITH DEPENDENT WEIGHTS

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1. Introduction[1]

The International Price Program (IPP) of the Bureau of Labor Statistics (BLS) measures aggregate price changes for samples of U.S. exporters and importers of agricultural goods, industrial supplies and materials, capital equipment and machinery, consumer merchandise, and transportation services. These measures are used by the U.S. Department of Commerce to adjust the monthly international trade figures and the quarterly National Income and Product Accounts for inflation or deflation (BLS 1997). These adjustments, also known as deflators, take the form of price indexes of the Laspeyres type, which means that as prices change over time, quantities are left at their original level determined at a base period. Estimating the variances of price indexes is desirable as a measure of accuracy and stability, but is difficult due to the nonlinearity of the function and in the IPP's case due to a complex sample design. Therefore several variance estimation techniques need to be examined and evaluated. The length of time from the "previous" period to the "current" period is important because some customers of IPP data are interested in short-term changes, such as monthly or quarterly percent changes, and some customers are interested in long-term changes, potentially of several years. The length of time chosen for IPP variance estimation is twelve months, which is useful to both short-term and long-term customers.

Section 2 describes the generation of a pseudo population from actual IPP data, and presents the models and their parameters from an economic approach, which places dependence of the estimation weights on the price levels. Section 3 shows the formulation of the target variance, to which the output of each variance estimation technique is compared. Section 4 summarizes the four techniques of variance estimation examined by the IPP. Section 5 gives the criteria used to compare the techniques and shows the results based on these criteria. Section 6 gives a recommendation for the International Price Program and other notable conclusions.

2. Pseudo Population

Four industries were chosen for the IPP variance estimation study: import and export platinum group metals, import and export textile machinery. For each industry 1000 pseudo establishments were generated and then detailed product areas, based on the Harmonized Classification System, were randomly assigned with replacement to the companies, using the frequency distribution of actual sampling frames made available by the Bureau of the Census for exports and by the Customs Service for imports. Each pairing of company and detailed product area was considered an instance of an item, so multiple random selections of the same pairing represented several items traded by the company within the same detailed product area.

The next step was to generate pseudo-population Laspeyres indexes and variances using models of base prices, base weights, and monthly price changes. The base prices were generated from a Gaussian distribution using IPP prices with outliers removed. Under the assumption that importers and exporters satisfy maximization behavior, the base weights, which are dependent on the base prices, use the Fisher model which applies Shephard's Lemma to calculate base quantities from the total share values and random coefficients of the unit cost function (Diewert and Nakamura 1993). The total share values were generated from a Gamma distribution and the coefficients were generated from a Uniform distribution. The monthly price changes were generated from a Double Gamma or Double Gaussian model (depending on the industry) with a spike at 0%.

The Laspeyres formula for a 12-month index is as follows:

$$I_t = \sum_{i=1}^{N} \sum_{j=1}^{M_i} w_{ijt} s_{ijt}$$

where I_t is the 12-month stratum[2] index ending at time t, N is the number of companies in the stratum, M_i is the number of items traded by company i,

[1] Opinions expressed in this paper are those of the author and do not constitute policy of the Bureau of Labor Statistics.

[2] A stratum is a broad category consisting of many items imported or exported by many U.S. companies.

$$w_{ijt} = \frac{x_{ij} r_{ij,t-12}}{\sum_{i=1}^{N} \sum_{j=1}^{M_i} x_{ij} r_{ij,t-12}}, \qquad s_{ijt} = \frac{r_{ijt}}{r_{ij,t-12}},$$

x_{ij} is the base period weight (price times quantity, summed over one year) for company i and item j, and r_{ijt} is the long-term relative (current price divided by average price over the base period) for company i and item j at time t.

The sample design used to select companies and items with which to estimate I_t is as follows:

1. Select n companies with probability proportional to size (Σx_{ij}), without replacement, sequentially with fixed order by descending size and one random start.
2. For each selected company, select m_i items using simple random sampling without replacement.

Let N_p be the number of companies available for selection in the stratum that are not selected with certainty, and let n_p be the number of companies actually selected from these N_p non-certainties. Note that N_p must be greater than one[3], n_p must be greater than zero, and $n_p < N_p$.

Let $\pi_i = \frac{n_p \sum_{j}^{M_i} x_{ij}}{\sum_{i}^{N_p} \sum_{j}^{M_i} x_{ij}}$ and

$$\alpha_i = \begin{cases} 1 \text{ if company } i \text{ selected} \\ 0 \text{ if company } i \text{ not selected} \end{cases};$$

then $E(\alpha_i) = \alpha_i$. Also let

$$\alpha_{ij} = \begin{cases} 1 \text{ if item } j \text{ selected within company } i \\ 0 \text{ if item } j \text{ not selected within company } i \end{cases};$$

then $E(\alpha_{ij}) = m_i / M_i$.

[3] The exception is when all N companies are selected with certainty, in which case $N_p = n_p = 0$.

The usual IPP estimator for I_t is as follows:

$$\hat{I}_t = \sum_{i=1}^{N} \sum_{j=1}^{M_i} \hat{w}_{ijt} s_{ijt} \quad \text{where}$$

$$\hat{w}_{ijt} = \frac{\dfrac{\alpha_i \alpha_{ij} x_{ij} r_{ij,t-12} M_i}{\pi_i m_i}}{\sum_{i=1}^{N} \sum_{j=1}^{M_i} \dfrac{\alpha_i \alpha_{ij} x_{ij} r_{ij,t-12} M_i}{\pi_i m_i}}.$$

Except for the familiar ratio bias (Särndal, Swensson, and Wretman 1992), $E(\hat{w}_{ijt}) = w_{ijt}$ and $E(\hat{I}_t) = I_t$.

To aggregate stratum indexes we need to add the subscript h to the index variable (I_{ht}). Then using H as the subscript of the parent stratum,

$$I_{Ht} = \sum_{h=1}^{L} W_{ht} I_{ht} ,$$

where $W_{ht} = \frac{\$_h r_{h,t-12}}{\sum_{h=1}^{L} \$_h r_{h,t-12}}$, L is the number of strata mapped to the parent stratum H, $\$_h$ is the base period weight for stratum h, and r_{ht} is the long-term relative for stratum h at time t.

3. Target Variance

For systematic sampling, the contribution of certainty companies to the first-stage variance component is zero, so the Yates-Grundy variance formula is as shown below (Wolter 1985):

$$V(\hat{I}_t) = \sum_{i=1}^{N_p-1} \sum_{i'=i+1}^{N_p} \left[(\pi_i \pi_{i'} - \pi_{ii'}) \left(\frac{\sum_{j=1}^{M_i} w_{ijt} s_{ijt}}{\pi_i} - \frac{\sum_{j=1}^{M_{i'}} w_{i'jt} s_{i'jt}}{\pi_{i'}} \right)^2 \right] + \sum_{i=1}^{N} \frac{M_i^2}{m_i \pi_i} \left(1 - \frac{m_i}{M_i}\right) \sigma_{wit}^2$$

where $\pi_{ii'}$ is the joint probability that companies i and i' are both selected,

and $\sigma_{wit}^2 = \frac{\sum_{j=1}^{M_i} w_{ijt}^2 s_{ijt}^2 - \frac{1}{M_i} \left(\sum_{j=1}^{M_i} w_{ijt} s_{ijt} \right)^2}{M_i - 1}$.

However, the ratio bias leads to a positive variance result even when all of the twelve-month price ratios (s_{ijt}) are the same. We therefore make an adjustment to reduce the ratio bias by replacing the $r_{ij,t-12}$ in the weights w_{ijt} with their straight average and by replacing the weighted σ_{wit}^2 with the unweighted version of the variance of short-term relatives. The target variance is therefore as follows:

$$V_{TRG}\left(\hat{I}_t\right)=\sum_{i=1}^{N_p-1}\sum_{i'=i+1}^{N_p}\left[\left(\pi_i\pi_{i'}-\pi_{ii'}\right)\left(\frac{\sum_{j=1}^{M_i}w_{ij}s_{ijt}}{\pi_i}-\frac{\sum_{j=1}^{M_{i'}}w_{i'j}s_{i'jt}}{\pi_{i'}}\right)^2\right]$$

$$+\sum_{i=1}^{N}\frac{M_i^2}{m_i\pi_i}\left(1-\frac{m_i}{M_i}\right)\sigma_{it}^2$$

where $w_{ij}=\dfrac{x_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{M_i}x_{ij}}$,

and $\sigma_{it}^2=\dfrac{\sum_{j=1}^{M_i}s_{ijt}^2-\frac{1}{M_i}\left(\sum_{j=1}^{M_i}s_{ijt}\right)^2}{M_i-1}$.

This target variance yields a zero result when all of the twelve-month price ratios are the same.

4. Variance Estimation Techniques

For each of the variance estimation techniques, the estimator for the aggregate stratum variance $V\left(\hat{I}_{Ht}\right)=\sum_{h=1}^{L}W_{ht}^2V_{TRG}\left(\hat{I}_{ht}\right)$ is calculated as follows:

$\hat{I}_{Ht}=\sum_{h=1}^{L}\hat{W}_{ht}\hat{I}_{ht}$, where $\hat{W}_{ht}=\dfrac{\$_h\hat{r}_{h,t-12}}{\sum_{h=1}^{L}\$_h\hat{r}_{h,t-12}}$,

and $\hat{r}_{ht}$ is the long-term relative estimate for stratum h at time t. So, $\hat{V}\left(\hat{I}_{Ht}\right)=\sum_{h=1}^{L}\hat{W}_{ht}^2\hat{V}\left(\hat{I}_{ht}\right)$.

When there are companies selected with certainty within a stratum h, these companies are considered separate variance estimation strata for some techniques. However, for the purpose of aggregate variance estimation, the variance due to the certainties should first be combined with the variance due to the non-certainties within each stratum h before aggregation of the variance estimates to parent stratum H.

4.1 Adjusted Ratio Biased Variance Estimator

The sample design used for this research matches the IPP products sample design at the first stage, but simplifies the second stage in order to use an adjusted Yates-Grundy target variance formula, shown in section 3. However, a condition of the Yates-Grundy variance estimator (Cochran 1977) is that $\pi_{ii'}$, the probability that companies i and i' are both in the sample, be not zero. This is also a condition for the Horvitz-Thompson variance estimator (Cochran 1977). These formulas give very biased estimates for the IPP sample design because the vast majority of the joint probabilities are zero.

Thus an alternative closed-form variance estimator was derived. This estimator assumes with-replacement sampling, which tends to have a higher variance than without-replacement sampling at the first stage. To handle this assumption we use the finite population correction factor for systematic pps sampling explained in Wolter 1985.

$$\hat{V}_{ARB}(\hat{I}_t)=\left[\sum_{i=1}^{n_p}\left(\frac{n_pM_i}{\pi_im_i}\sum_{j=1}^{m_i}u_{ij}s_{ijt}-\sum_{i=1}^{n_p}\frac{M_i}{\pi_im_i}\sum_{j=1}^{m_i}u_{ij}s_{ijt}\right)^2\right]\times\left(\frac{1}{n_p(n_p-1)}\right)\left(1-\frac{1}{n_p}\sum_{i=1}^{n_p}\pi_i\right)$$

where $u_{ij}=\dfrac{x_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{m_i}\frac{x_{ij}M_i}{\pi_im_i}}$.

4.2 Taylor Linearized Variance Estimator

The Taylor Linearized Variance Estimator (Hansen, Hurwitz, & Madow 1953) replaces the complexity of estimation of a ratio by a difference of linear estimators. In order to compensate for the natural under-estimation due to neglecting higher order terms of the Taylor approximation, the choice of sample design is probability proportionate to size with replacement even though the actual sample selection was without replacement (Bureau of the Census 1993).

$$\hat{V}_{TLWR}\left(\hat{I}_t\right)=\hat{I}_t^2\ \frac{n}{n-1}\sum_{i=1}^{n}\left[\sum_{j=1}^{m_i}v_{ij}\left(r_{ijt}\Big/\hat{Y}_t-r_{ij,t-12}\Big/\hat{Y}_{t-12}\right)\right]^2$$

where $\hat{Y}_t=\sum_{i=1}^{n}\sum_{j=1}^{m_i}v_{ij}r_{ijt}$ and $v_{ij}=\dfrac{x_{ij}M_i}{\pi_i m_i}$.

4.3 Stratified Jackknife Variance Estimator

Here we first obtain subsample index estimates by deleting observations from the sample. Then an estimate of the variance for the full sample is found from the variability of these subsample estimates about the full sample estimate (Wolter 1985).

First we divide the stratum up into substrata ℓ and rewrite the estimator of Laspeyres revenue (Y_t) as $\hat{Y}_t=\sum_{\ell=1}^{L}\sum_{i=1}^{n_\ell}\sum_{j=1}^{m_{\ell i}}v_{\ell ij}r_{ijt}$ where L is the total number of substrata; n_ℓ is the number of sample companies in substratum ℓ; $m_{\ell i}$ is the number of sample items in company *i* in substratum ℓ; and $v_{\ell ij}$ is the sampling weight for substratum ℓ, company *i*, item *j* (actually the same as v_{ij} except that a subscript ℓ has been inserted for the substratum).

Let λ be the substratum being jackknifed and α the company in substratum λ being jackknifed. The jackknifed long term relative is given by

$$\hat{Y}_{t\lambda\alpha}=\sum_{\substack{i=1\\ i\neq\alpha}}^{n_\lambda}\sum_{j=1}^{m_{\lambda i}}v_{\lambda ij}r_{ijt}\left(\frac{n_\lambda}{n_\lambda-1}\right)+\sum_{\substack{\ell=1\\ \ell\neq\lambda}}^{L}\sum_{i=1}^{n_\ell}\sum_{j=1}^{m_{\ell i}}v_{\ell ij}r_{ijt}$$

(Fay 1995). This relative is computed as usual, except with element α eliminated from substratum λ and with a weight factor to compensate for the missing unit. The jackknifed 12-month index is given by $\hat{I}_{t\lambda\alpha}=\hat{Y}_{t\lambda\alpha}\big/\hat{Y}_{t-12,\lambda\alpha}$. The full 12-month index is

$$\hat{I}_t=\sum_{\ell=1}^{L}\sum_{i=1}^{n_\ell}\sum_{j=1}^{m_{\ell i}}v_{\ell ij}r_{ijt}\Bigg/\sum_{\ell=1}^{L}\sum_{i=1}^{n_\ell}\sum_{j=1}^{m_{\ell i}}v_{\ell ij}r_{ij,t-12}$$
$$=\hat{Y}_t\big/\hat{Y}_{t-12}$$

and the stratified jackknife variance estimator is

$$\hat{V}_{STJ}\left(\hat{I}_t\right)=\sum_{\lambda=1}^{L}\left(1-\frac{1}{n_\lambda}\sum_{i=1}^{n_\lambda}\pi_i\right)\left(\frac{n_\lambda-1}{n_\lambda}\right)\sum_{\alpha=1}^{n_\lambda}\left(\hat{I}_{t\lambda\alpha}-\hat{I}_t\right)^2$$

(Wolter 1985).

4.4 Stratified Systematically Grouped Variance Estimator

For this technique groups are formed so that each group has essentially the same sampling design as the parent sample. We chose ten groups (see Fay 1995 for tips on choosing the number of groups), systematically rather than randomly because the sample design is sequential with a fixed order. We continue to define n_ℓ as the number of sample companies in substratum ℓ and now also define $n_{\ell g}$ to be the number of sample companies in substratum ℓ and systematic group *g*.

We write the replicate estimate of the Laspeyres revenue for each systematic group *g* within each substratum ℓ:

$$\hat{Y}_{\ell gt}=\left(\frac{n_\ell}{n_{\ell g}}\right)\sum_{i=1}^{n_{\ell g}}v_{\ell i}r_{\ell it}\ ,$$

and the replicate estimate for the full stratum:

$$\hat{Y}_{gt}=\hat{Y}_t+\sum_{\ell=1}^{L}\left[\sqrt{1-\frac{1}{n_\ell}\sum_{i=1}^{n_\ell}\pi_i}\left(\hat{Y}_{\ell gt}-\hat{Y}_{\ell t}\right)\right],$$

where $\hat{Y}_{\ell t}=\sum_{i=1}^{n_\ell}v_{\ell i}r_{\ell it}$ is the Laspeyres revenue estimate for substratum ℓ. The replicate sample index is given by $\hat{I}_{gt}=\dfrac{\hat{Y}_{gt}}{\hat{Y}_{g,t-12}}$, and the stratified systematically grouped variance estimator is

$$\hat{V}_{SSG}\left(\hat{I}_t\right)=\frac{1}{90}\sum_{g=1}^{10}\left(\hat{I}_{gt}-\hat{I}_t\right)^2\ .$$

5. Results

For each of the four chosen industries, Chi-Square and Kolmogorov-Smirnov goodness of fit tests were used on actual IPP prices to determine whether the Student's T, the Gaussian, or the Gamma distribution was best for generation of monthly price changes. For platinum group metals the Gamma distribution was selected for both imports and exports. For textile machinery the Gamma distribution was selected for imports and the Gaussian for exports.

S-PLUS programs were written to simulate 35 populations for each of the four industries. Then SAS programs were written to calculate population variances, to select 50 samples from each population, and for each technique to calculate variance estimates. The comparison of the estimates to the population values used standard deviations instead of variances in order to bring the scale closer to one. Using the 50 samples for a given population, the mean squared error (MSE) of standard deviation estimates was calculated. The following tables show the results, comparing the different variance estimation methods across the four industries. Table 1 gives the MSE along with the comparative variance and bias squared percentages, while Table 2 gives the coefficients of variation (CV).

Table 1: Mean Squared Errors of the Standard Deviation Estimates

	Adj Ratio Biased		
Industry	**MSE**	**%Var**	**%B2d**
Import platinum group metals	0.000026	81.09%	18.91%
Export platinum group metals	0.000016	72.79%	27.21%
Import textile machinery	0.000014	29.96%	70.04%
Export textile machinery	0.000094	3.51%	96.49%

	Taylor Linearized		
Industry	**MSE**	**%Var**	**%B2d**
Import platinum group metals	0.000048	70.75%	29.25%
Export platinum group metals	0.000024	72.09%	27.91%
Import textile machinery	0.000005	74.05%	25.95%
Export textile machinery	0.000006	78.02%	21.98%

	Stratified Jackknife		
Industry	**MSE**	**%Var**	**%B2d**
Import platinum group metals	0.000040	75.53%	24.47%
Export platinum group metals	0.000021	76.15%	23.85%
Import textile machinery	0.000008	48.36%	51.64%
Export textile machinery	0.000009	47.24%	52.76%

	Stratified Systematically Grouped		
Industry	**MSE**	**%Var**	**%B2d**
Import platinum group metals	0.000129	36.59%	63.41%
Export platinum group metals	0.000113	25.25%	74.75%
Import textile machinery	0.000091	0.22%	99.78%
Export textile machinery	0.000069	0.25%	99.75%

Table 2: Coefficients of Variation for the Standard Deviation Estimates

	Adj Ratio Biased	Taylor Linearized
Import platinum group metals	14.68%	18.81%
Export platinum group metals	13.04%	15.59%
Import textile machinery	26.63%	17.27%
Export textile machinery	53.67%	22.02%

	Stratified Jackknife	Stratified Systematically Grouped
Import platinum group metals	17.88%	43.09%
Export platinum group metals	15.06%	49.37%
Import textile machinery	27.06%	333.45%
Export textile machinery	33.51%	371.77%

In addition, coverage rates and average successful interval lengths (ASIL) were calculated using 200 samples. These statistics do not compare point estimates to population values, as the MSE and CV do, but show how often the 95% confidence interval contains the sample mean and how long the successful intervals are on the average. Given the same Coverage Rate, tighter confidence intervals are better.

Table 3: Coverage Rates for the Standard Deviation Estimates

	Adj Ratio Biased	Taylor Linearized
Import platinum group metals	100.0%	100.0%
Export platinum group metals	100.0%	100.0%
Import textile machinery	79.5%	100.0%
Export textile machinery	41.5%	99.5%

	Stratified Jackknife	Stratified Systematically Grouped
Import platinum group metals	100.0%	100.0%
Export platinum group metals	100.0%	99.5%
Import textile machinery	99.5%	96.0%
Export textile machinery	100.0%	97.0%

Table 4 Average Successful Interval Lengths for the Standard Deviation Estimates

	Adj Ratio Biased	Taylor Linearized
Import platinum group metals	.016524	.019429
Export platinum group metals	.013189	.013785
Import textile machinery	.006691	.007070
Export textile machinery	.006835	.007939

	Stratified Jackknife	Stratified Systematically Grouped
Import platinum group metals	.019008	.025043
Export platinum group metals	.013102	.019931
Import textile machinery	.007448	.002853
Export textile machinery	.007635	.002413

6. Conclusions

The Stratified Systematically Grouped method gives reasonable ASIL, but shows the worst MSE and CV. The Adjusted Ratio Biased method gives poor coverage rates for the textile machinery industries. The Taylor Linearized and Stratified Jackknife methods give reasonable results that are similar to each other. The CV for the textile machinery industries were moderately better for Taylor Linearization than for Stratified Jackknife. Also due to its relative computational simplicity, the recommended technique for calculation of 12-month index variance estimates for the IPP is the Taylor Linearization method.

The next step for this research within IPP is the baseline calculation of variance estimates for all import and export industries. Also, estimates of the ratio bias on actual IPP indexes can be calculated, or at least bounded (Särndal, Swensson, and Wretman 1992), by the following:

$$Bias^2 \le \hat{V}\left[\hat{I}_t\right]\cdot\hat{V}\left[\hat{Y}_{t-12}\right]/\left[\hat{Y}_{t-12}\right]^2 .$$

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