DISTRIBUTIONS OF RATIOS FOR SOME SELECTED BIVARIATE PROBABILITY FUNCTIONS

Paul W. Mielke, Jr., Colorado State University
John A. Flueck, Temple University

I. INTRODUCTION

Ratio estimators, R = Y/X of random variables (r.v.'s) X and Y, of population ratios have long been utilized in both theoretical and applied research. Examples from the theory of statistics include ratios of sample means in survey sampling, ratios of mean squares in experimental design, and likelihood ratios in testing. In the experimental areas, examples include crime indices in law enforcement, inheritance ratios in biology, effectiveness ratios in psychology, unemployment ratios in economics, stress ratios in engineering, precipitation ratios in meteorology, and survival and other ratios in demography.

However, despite this broad utilization, the distributional properties of the ratio R, generally have not been well developed. As Cochran (1963) has stated, "The distribution of the ratio estimate has proved annoyingly intractable because both X and Y vary from sample to sample. The known theoretical results fall short of what we would like to know for practical applications."

In this paper we will attempt to improve on this situation by reviewing some previously published literature on distributions of ratios and by presenting distributional and moment results for the ratio of two r.v.'s X and Y under selected bivariate structures (i.e., gamma, generalized gamma, beta of the first kind, log-normal, Weibull, beta-P, and beta-κ).

Greenwood and White (1910) initiated work on the distribution of R in their work on the opsonic index. They empirically generated sampling distributions of the opsonic index (ratio) under the assumption that X and Y are statistically independent. They noted that the resulting empirical distribution was skewed and had a mean >1.0. Pearson (1910), in a pioneering companion paper, presented Greenwood's initial theoretical effort along with the moments for R in the more general case where X and Y were correlated. However, he felt that "these approximate formulae would be practically unworkable if X and Y were correlated...." and hence concerned himself with an examination of the uncorrelated case. He concluded that (1) "if the distribution of both X and Y be symmetrical..., the distribution of indices must be skew" and (2) "the mean of the ratio of two numbers picked out of the same series is certainly greater than unity if the series be symmetrical, and will probably be always greater than unity even if it be not." Pearson went on to estimate the distribution of the opsonic index, for X and Y statistically independent, utilizing estimated moments to select and fit one of the Pearsonian type distributions. These estimated distributions were typically in close agreement with their empirical counterparts presented by Greenwood and White.

For the normal distribution, Merrill (1928) considered the ratio of two correlated normals using the method of moments and concluded "that for conditions ordinarily met in practice, the frequency distribution of the index, when both components follow closely the normal law, is sensibly normal." However, he also noted that "in cases where the correlation is high and the coefficients of variation are large, there may be a considerable deviation from normality." Craig (1929) presented a method of describing the distribution of R in terms of its semi-invariants (functions of the estimated moments of the joint distribution of X and Y with bounding restriction on X) and correctly indicated the persistent lack of normality of R and the condition needed for symmetry of its distribution. Geary (1930) followed with a "standardizing" of the ratio function (b+Y)/(a+X) [X and Y jointly distributed as $N(\mu_X,\mu_Y,\sigma_X,\sigma_Y,\rho_{X,Y})$] and showed it was approximately distributed $N(0,1)$, provided $a+X>0$. He also presented the exact probability density function (pdf) of this ratio function for a=b=0, and (although not recognizing it as a member of the Cauchy family) commented on its nonnormality and infinite moments. Fieller (1932) presented and exact general solution to the distribution of the ratio of two jointly distributed normals with means μ_1 and μ_2, standard deviations σ_1 and σ_2, and correlation coefficient ρ. He also showed how Geary's and Merrill's results could be achieved under the assumption of a truncated normal distribution. Rao (1952) proved for a wide class of estimators (including R) that the standardized form was asymptotically distributed as N(0,1). Marsaglia (1965) presented the results of a study of the ratio of correlated normal r.v.'s utilizing computer drawn graphs of the pdf of R (producing symmetrical, nonsymmetrical unimodal, and bimodal pdf's). Press (1969) presented a similar study for correlated student's t r.v.'s. Rao and Garg (1969) presented the ratio of powers of the absolute value of two independent standardized normals and found the result to be generalized positive Cauchy distribution (a special case of the generalized beta of the second kind).

The ratio of two gamma r.v.'s was initiated by Kullback (1936) for the ratio of two independent gammas (i.e. a beta of the second kind). He also showed that Fisher's (1924) result for the distribution of the log of the ratio of two independent sample variances was a special case. Recently, Flueck, Holland, and Lee (1975) presented exact results for the density function of the ratio of two correlated gamma r.v.'s and illustrated some of the resulting properties of R with sixty-four computer drawn graphs.

The chi density function is closely related to the gamma, and Bose (1935) presented the density function for the ratio of two correlated chi r.v.'s. Krishnaiah, Hagis, and Steinberg (1963) presented the t^{th} moments of this ratio and extensive tables of its cumulative distribution function.

Rietz (1939) graphically presented pdf's for a special case of the ratio of two betas of the first kind (i.e., uniform r.v.'s given particular uncorrelated and correlated structure). Broadbent (1954) extended these results in presenting the ratio of both the uniform and triangular r.v.'s to an arbitrary independent positive r.v. Marsaglia (1965) presented results for the pdf of the ratio of sums of independent uniform r.v.'s and, even with relatively few terms, the ratio was quite closely approximated by the ratio of two independent normal r.v.'s.

The ratio of two independent log-normals, as well as the ratio of products, is given by Aitchison and Brown (1966). The resulting distributions are naturally log-normal.

Malik (1967) generalized Kullback's result for two independent gammas by deriving the density function for the ratio of two independent generalized gamma r.v.'s. Rao and Garg (1969) have indicated that the resulting pdf is a member of the family of generalized beta distributions of the second kind. Block and Rao (1973) subsequently generalized Malik's result by presenting the density and moments of the ratio of two independent distended gamma r.v.'s. The resulting density function was termed the distended beta.

In summary, the above brief history indicates that the distribution of the ratio of normal, student's t, gamma, chi, and a special case of the beta of the first kind r.v.'s has been presented for correlated X and Y.

II. SOME NEW RESULTS

The distribution of R for X and Y independent is conceptually attainable (e.g., Huntington, 1939). The situation when X and Y are correlated is more complex.

A. Gamma

Let X, Y and P be independent gamma r.v.'s with common scale parameter A and shape parameters $\alpha-\phi$, $\beta-\phi$ and ϕ, respectively $\{0\le\phi<\min(\alpha,\beta)\}$. As suggested by Cherian (1941) and David and Fix (1961), following Weldon, if U=X+P and V=Y+P, then the bivariate pdf of the r.v.'s U and V may be expressed as

$$f_{U,V}(u,v) = \frac{e^{-u-v}}{\Gamma(\alpha-\phi)\Gamma(\beta-\phi)\Gamma(\phi)} \cdot \int_0^{\min(u,v)} p^{\phi-1}(u-p)^{\alpha-\phi-1}(v-p)^{\beta-\phi-1}e^{p}dp, \quad (2.1)$$

where A is assumed equal to unity because it enters in the same form in both numerator and denominator of the ratio. It can be shown that equation (2.1) is equivalent to

$$f_{U,V}(u,v) = \begin{cases} \dfrac{u^{\alpha-1}v^{\beta-\phi-1}e^{-(u+v)}}{\Gamma(\alpha)\Gamma(\beta-\phi)} \cdot F_1^*(\phi,1+\phi-\beta,\alpha;\frac{u}{v},-u), & 0<u\le v, \\ \dfrac{u^{\alpha-\phi-1}v^{\beta-1}e^{-(u+v)}}{\Gamma(\alpha-\phi)\Gamma(\beta)} \cdot F_1^*(\phi,1+\phi-\alpha,\beta;\frac{v}{u},-v), & 0<v<u, \end{cases} \quad (2.2)$$

where

$$F_1^*(a,b,c;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_n}{(c)_{m+n}m!n!} x^m y^n, \quad |x|<1$$

is a "degenerate" two variable hypergeometric function (Gradshteyn and Ryzhik, 1965, p. 1067) and $(a)_b=\Gamma(a+b)/\Gamma(a)$. Thus, from the reproductive property of the gamma, U and V are gamma r.v.'s with marginal distribution shape parameters α and β, respectively, and dependence parameter ϕ. If $\phi=0$, then U and V are independent r.v.'s.

The nonnegative integral moments of U and V are given by

$$E(U^sV^t) = \sum_{j=0}^{s}\sum_{k=0}^{t} \binom{s}{j}\binom{t}{k}(\alpha-\phi)_{s-j}(\beta-\phi)_{t-k}(\phi)_{j+k}.$$

In particular, $E(U)=Var(U)=\alpha$, $E(V)=Var(V)=\beta$, $Cov(U,V)=\phi$, and $\rho(U,V)=\phi(\alpha\beta)^{-1/2}$. Let the ratio of the two gamma r.v.'s be defined as R=U/V. Then following the approach of Flueck and Holland (1974) and using a change of variable, the pdf of R is

$$f_R(r) = \begin{cases} \dfrac{r^{\alpha-1}(1+r)^{\phi-\alpha-\beta}}{B(\alpha,\beta-\phi)} F_1(\phi,\alpha+\beta-\phi,1+\phi-\beta,\alpha;\frac{r}{1+r},r), & 0<r\le 1, \\ \dfrac{r^{\alpha-\phi-1}(1+r)^{\phi-\alpha-\beta}}{B(\alpha-\phi,\beta)} \cdot F_1(\phi,\alpha+\beta-\phi,1+\phi-\alpha,\beta;\frac{1}{1+r},\frac{1}{r}), & 1<r, \end{cases} \quad (2.3)$$

where

$$F_1(a,b,c,d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}m!n!} x^m y^n, \quad |x|<1,\ |y|<1$$

is a two variable hypergeometric function (Gradshteyn and Ryzhik, 1965, p. 1053) and

$$B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b).$$

A convenient computational form has been presented by Flueck, Holland and Lee (1975).

The integral moments of R are given by

$$E(R^s) = \begin{cases} \sum_{j=0}^{s}\binom{s}{j}\dfrac{(\alpha-\phi)_j(\phi)_{s-j}}{(\beta-j)_s}, & 0\le s, \\ \sum_{j=0}^{-s}\binom{-s}{j}\dfrac{(\beta-\phi)_j(\phi)_{-s-j}}{(\alpha-j)_{-s}}, & s<0. \end{cases}$$

In particular,

$$E(R) = \frac{\alpha\beta-\phi}{\beta(\beta-1)}, \quad 1<\beta, \quad \text{and}$$

$$Var(R)=\{\alpha\beta^2[(\alpha+\beta)\beta+\alpha-1]-2\beta[\alpha\beta(\beta+1)+(2\beta-1)(\beta-1)]\phi + [5\beta(\beta-1)+2]\phi^2\}/[(\beta+1)\beta^2(\beta-1)^2(\beta-2)], \quad 2<\beta.$$

If $(U_1,V_1),\ldots,(U_n,V_n)$ are n independent pairs from (2.2), then the analogous results for sums of gamma r.v.'s,

$$U^*=\sum_{i=1}^{n}U_i \text{ and } V^*=\sum_{i=1}^{n}V_i$$

can be shown to follow when the parameter set (α,β,ϕ) is replaced with $(n\alpha,n\beta,n\phi)$. As a consequence of these results and a general result by Rao (1952, p. 207), it follows that the asymptotic distribution of $R^*=U^*/V^*$ is normal with mean α/β and variance $\alpha(\alpha+\beta-2\phi)/n\beta$.

Using Stacy's (1962) generalization of the gamma, the previous results can be further generalized by constructing a bivariate pdf of generalized gamma r.v.'s given by

$$f_{W,Z}(w,z)=f_{U,V}(w^{\gamma},z^{\delta})\gamma\delta w^{\gamma-1}z^{\delta-1}$$

where $W=U^{1/\gamma}$, $Z=V^{1/\delta}$, $0 < \gamma$ and $0 < \delta$. If $\gamma=\delta$, then the pdf of the ratio r.v. $Q=W/Z$ is

$$f_Q(q)=f_R(q^{\gamma})\gamma q^{\gamma-1}.$$

B. Beta of the First Kind

Employing an approach similar to that of the previous section yields two distinctly different cases for a bivariate beta distribution of the first kind. The parameterizations used here follow Mielke (1975).

Case 1:

Let X, Y and P be independent beta r.v.'s with shape parameters $[p\gamma,(1-p)\gamma-\phi]$, $[q\gamma,(1-q)\gamma-\phi]$ and $(\gamma-\phi,\phi)$, respectively $\{0<p<1,\ 0<q<1,\ 0<\gamma,\ 0\le\phi<\min[(1-p)\gamma,(1-q)\gamma]\}$. If $U = XP$ and $V = YP$, then the bivariate pdf of U and V may be shown to be

$$f_{U,V}(u,v) = \begin{cases} \dfrac{u^{p\gamma-1}(1-u)^{(1-p)\gamma-\phi-1}v^{q\gamma-1}(1-v)^{(1-q)\gamma-1}}{B[p\gamma,(1-p)\gamma-\phi]B[q\gamma,(1-q)\gamma]} \\ \cdot F_1[\phi,1-(1-p)\gamma+\phi,\gamma-\phi-1,(1-q)\gamma;\frac{1-v}{1-u},1-v], \\ 0 < u \le v < 1, \\ \dfrac{u^{p\gamma-1}(1-u)^{(1-p)\gamma-1}v^{q\gamma-1}(1-v)^{(1-q)\gamma-\phi-1}}{B[p\gamma,(1-p)\gamma]B[q\gamma,(1-q)\gamma-\phi]} \\ \cdot F_1[\phi,1-(1-q)\gamma+\phi,\gamma-\phi-1,(1-p)\gamma;\frac{1-u}{1-v},1-u], \\ 0 < v < u < 1. \end{cases} \quad (2.4)$$

Here U and V are beta r.v.'s with marginal distribution shape parameters $[p\gamma,(1-p)\gamma]$ and $[q\gamma,(1-q)\gamma]$, respectively, and dependence parameter ϕ. If $\phi=0$, then U and V are independent r.v.'s.

The moments of U and V are given by

$$E(U^sV^t)= \frac{(p\gamma)_s(q\gamma)_t(\gamma-\phi+t)_s}{(\gamma)_{s+t}(\gamma-\phi)_s} = \frac{(p\gamma)_s(q\gamma)_t(\gamma-\phi+s)_t}{(\gamma)_{s+t}(\gamma-\phi)_t}.$$

In particular, $E(U)=p$, $\mathrm{Var}(U)=\frac{p(1-p)}{\gamma+1}$, $E(V)=q$, $\mathrm{Var}(V)=\frac{q(1-q)}{\gamma+1}$, $\mathrm{Cov}(U,V)= \frac{pq\phi}{(\gamma-\phi)(\gamma+1)}$, and $\rho(U,V)=\frac{\phi}{\gamma-\phi}\left[\frac{pq}{(1-p)(1-q)}\right]^{1/2}$. If $R=U/V$, the identity $U/V=X/Y$ conveniently yields the pdf of R given by

$$f_R(r) = \begin{cases} \dfrac{r^{p\gamma-1}B[(p+q)\gamma,(1-q)\gamma-\phi]}{B[p\gamma,(1-p)\gamma-\phi]B[q\gamma,(1-q)\gamma-\phi]} \\ \cdot F[1-(1-p)\gamma+\phi,(p+q)\gamma;(1+p)\gamma-\phi;r], \\ 0 < r \le 1, \\ \dfrac{r^{-q\gamma-1}B[(p+q)\gamma,(1-p)\gamma-\phi]}{B[p\gamma,(1-p)\gamma-\phi]B[q\gamma,(1-q)\gamma-\phi]} \\ \cdot F[1-(1-q)\gamma+\phi,(p+q)\gamma;(1+q)\gamma-\phi;\frac{1}{r}], \\ 1 < r. \end{cases} \quad (2.5)$$

where

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

is the Gaussian hypergeometric function of one variable (Gradshteyn and Ryzhik, 1965, p. 1039). The moments of R are given by

$E(R^s) = \frac{(p\gamma)_s(q\gamma)_{-s}}{(\gamma-\phi)_s(\gamma-\phi)_{-s}}$. In particular,

$E(R) = \frac{p\gamma(\gamma-\phi-1)}{(\gamma-\phi)(q\gamma-1)}$, $1<q\gamma$, and

$$\mathrm{Var}(R)=p\gamma(\gamma-\phi-1)[(\gamma-\phi)^2(\gamma-\phi+1)(q\gamma-2)(q\gamma-1)^2]^{-1} \cdot\{p\gamma(\gamma-\phi-1)[(1-q)\gamma-\phi]+[(1-p)\gamma-\phi](\gamma-\phi-2)(q\gamma-1)\},$$
$2 < q\gamma$.

Case 2:

Let X, Y and P be independent beta r.v.'s with paired shape parameters $[pq\gamma+\phi,(1-p)q\gamma-\phi]$, $[qp\gamma+\phi,(1-q)p\gamma-\phi]$ and $(pq\gamma,\phi)$, respectively $\{0<p<1,0<q<1,0<\gamma,0\le\phi<\min[(1-p)q\gamma,(1-q)p\gamma]\}$. If $U=XP$ and $V=YP$, then the bivariate pdf of U and V may be expressed as

$$f_{U,V}(u,v) = \begin{cases} \dfrac{u^{pq\gamma+\phi-1}(1-u)^{(1-p)q\gamma-\phi-1}v^{qp\gamma+\phi-1}(1-v)^{(1-q)p\gamma-1}}{B[pq\gamma+\phi,(1-p)q\gamma-\phi]B[qp\gamma+\phi,(1-q)p\gamma]} \\ \cdot F_1[\phi,1+\phi-(1-p)q\gamma,(p+q-pq)\gamma-1,(1-q)p\gamma;\frac{1-v}{1-u},1-v], \\ 0 < u \le v < 1, \\ \dfrac{u^{pq\gamma+\phi-1}(1-u)^{(1-p)q\gamma-1}v^{qp\gamma+\phi-1}(1-v)^{(1-q)p\gamma-\phi-1}}{B[pq\gamma+\phi,(1-p)q\gamma]B[qp\gamma+\phi,(1-q)p\gamma-\phi]} \\ \cdot F_1[\phi,1+\phi-(1-q)p\gamma,(p+q-pq)\gamma-1,(1-p)q\gamma;\frac{1-u}{1-v},1-u], \\ 0 < v < u < 1. \end{cases} \quad (2.6)$$

Now U and V are beta r.v.'s with marginal distribution shape parameters $[pq\gamma,(1-p)q\gamma]$ and $[qp\gamma,(1-q)p\gamma]$, respectively, and dependence parameter ϕ. Again, if $\phi = 0$, then U and V are independent r.v.'s.

The moments of U and V are given by

$$E(U^sV^t)= \frac{(pq\gamma)_{s+t}(pq\gamma+\phi)_s}{(q\gamma)_s(p\gamma)_t(pq\gamma+\phi+t)_s} = \frac{(pq\gamma)_{s+t}(pq\gamma+\phi)_t}{(q\gamma)_s(p\gamma)_t(pq\gamma+\phi+s)_t}.$$

In particular, $E(U) = p$, $\mathrm{Var}(U) = \frac{p(1-p)}{q\gamma+1}$, $E(V) = q$, $\mathrm{Var}(V) = \frac{q(1-q)}{p\gamma+1}$, $\mathrm{Cov}(U,V) = \frac{\phi}{\gamma(pq\gamma+\phi+1)}$, and $\rho(U,V) = \frac{\phi}{\gamma(pq\gamma+\phi+1)}\left[\frac{(p\gamma+1)(q\gamma+1)}{p(1-p)q(1-q)}\right]^{1/2}$.

If $R = U/V$, the identity $U/V = X/Y$ again yields the pdf of R given by

$$f_R(r) = \begin{cases} \dfrac{r^{pq\gamma+\phi-1}B[2pq\gamma+2\phi,(1-q)p\gamma-\phi]}{B[pq\gamma+\phi,(1-p)q\gamma-\phi]B[qp\gamma+\phi,(1-q)p\gamma-\phi]} \\ \cdot F[1+\phi-(1-p)q\gamma,2pq\gamma+2\phi;(1+q)p\gamma+\phi;r], \\ 0 < r \le 1, \\ \dfrac{r^{-pq\gamma-\phi-1}B[2pq\gamma+2\phi,(1-p)q\gamma-\phi]}{B[pq\gamma+\phi,(1-p)q\gamma-\phi]B[qp\gamma+\phi,(1-q)p\gamma-\phi]} \\ \cdot F[1+\phi-(1-q)p\gamma,2pq\gamma+2\phi;(1+p)q\gamma+\phi;\frac{1}{r}], \\ 1 < r. \end{cases} \quad (2.7)$$

The moments of R are given by

$E(R^s)=\frac{(pq\gamma+\phi)_s(pq\gamma+\phi)_{-s}}{(q\gamma)_s(p\gamma)_{-s}}$. In particular,

$E(R)=\frac{(pq\gamma+\phi)(p\gamma-1)}{q\gamma(pq\gamma+\phi-1)}$, $1<pq\gamma+\phi$, and

$Var(R)=(pq\gamma+\phi)(p\gamma-1)$
$\cdot\{q^2\gamma^2(q\gamma+1)(pq\gamma+\phi-2)(pq\gamma+\phi-1)^2\}^{-1}$
$\cdot\{(p\gamma-1)(pq\gamma+\phi-2)[(1-p)q\gamma-\phi]$
$+q\gamma(pq\gamma+\phi+1)[(1-q)p\gamma-\phi]\}$, $2<pq\gamma+\phi$.

C. Log-Normal

Let X and Y be dependent log-normal r.v.'s with marginal distribution shape parameters α and β, and scale parameters A and B, respectively. For brevity, let $U=X/A$ and $V=Y/B$. Then the bivariate pdf of U and V is

$$f_{U,V}(u,v) = \frac{1}{2\pi uv\alpha\beta(1-\phi^2)^{1/2}} \cdot e^{-\frac{1}{2(1-\phi^2)}\left[\left(\frac{\ell n u}{\alpha}\right)^2-2\phi\left(\frac{\ell n u}{\alpha}\right)\left(\frac{\ell n v}{\beta}\right)+\left(\frac{\ell n v}{\beta}\right)^2\right]},$$
$$0 < \min(u,v), \qquad (2.8)$$

with dependence parameter $\phi\{0 < \min(\alpha,\beta), |\phi| < 1\}$. If $\phi=0$, then U and V are independent r.v.'s.

The moments of U and V are given by $E(U^sV^t)=e^{(s^2\alpha^2+t^2\beta^2+2\phi st\alpha\beta)/2}$. In particular, $E(U) = e^{\alpha^2/2}$, $Var(U)=e^{\alpha^2}(e^{\alpha^2}-1)$, $E(V)=e^{\beta^2/2}$, $Var(V)=e^{\beta^2}(e^{\beta^2}-1)$, $Cov(U,V)=e^{(\alpha^2+\beta^2)/2}(e^{\phi\alpha\beta}-1)$, and $\rho(U,V)=(e^{\phi\alpha\beta}-1)\{(e^{\alpha^2}-1)(e^{\beta^2}-1)\}^{-1/2}$. If $R=U/V$, then (2.8) yields the pdf of R given by

$$f_R(r)= \frac{1}{r\{2\pi(\alpha^2+\beta^2-2\phi\alpha\beta)\}^{1/2}} e^{-\frac{(\ell n r)^2}{2(\alpha^2+\beta^2-2\phi\alpha\beta)}},$$
$$0 < r. \qquad (2.9)$$

The moments of R are given by $E(R^s)=e^{s^2(\alpha^2+\beta^2-2\phi\alpha\beta)/2}$. In particular, $E(R)=e^{(\alpha^2+\beta^2-2\phi\alpha\beta)/2}$, and $Var(R)=e^{\alpha^2+\beta^2-2\phi\alpha\beta}(e^{\alpha^2+\beta^2-2\phi\alpha\beta}-1)$.

If $(U_1,V_1),\ldots,(U_n,V_n)$ are n independent pairs from (2.8), then the analogous results for n^{th} roots of the products of log-normal r.v.'s

$$U^*=\left(\prod_{i=1}^{n} U_i\right)^{1/n} \text{ and } V^*=\left(\prod_{i=1}^{n} V_i\right)^{1/n}$$

follow when the parameter set (α,β,ϕ) is replaced with $(n^{-1/2}\alpha,n^{-1/2}\beta,\phi)$. A consequence of these results and a property of the log-normal distribution is that the asymptotic distribution of $R^*=U^*/V^*$ is normal with mean 1 and variance $(\alpha^2+\beta^2-2\phi\alpha\beta)/n$.

D. Weibull

The Weibull, beta-P, and beta-κ r.v.'s (the last two r.v.'s being restricted generalized beta distribution r.v.'s of the second kind, Mielke and Johnson, 1974), utilize a generalization of results presented by Farlie (1960).

Let X^* and Y^* be r.v.'s having closed form marginal cumulative distribution functions (CDF's). The scale parameters of X^* and Y^* will be eliminated by the simple transformations, $U=X^*/A$ and $V=Y^*/B$, in order to achieve brevity in exposition. Thus U and V are dependent r.v.'s having closed form marginal CDF's $x=F_U(u)$ and $y=F_V(v)$.

The joint CDF's of U and V can be expressed as $F_{U,V}(u,v)=xy\{1-g(x,y)\}$. Since $0\le F_{UV}(u,v)\le 1$, $g(x,y)\le 1\le g(x,y)+(xy)^{-1}$. The existence of $\frac{\partial g(x,y)}{\partial x}$, $\frac{\partial g(x,y)}{\partial y}$ and $\frac{\partial^2 g(x,y)}{\partial x\partial y}$ is assumed. Also, since $F_{U,V}(\infty,v)=F_V(v)$ and $F_{U,V}(u,\infty)=F_U(u)$, then $g(1,y)=g(x,1)=0$. The monotonicity of $F_{U,V}(u,v)$ implies $\frac{\partial}{\partial x}[xy\{1-g(x,y)\}]\ge 0$ and $\frac{\partial}{\partial y}[xy\{1-g(x,y)\}]\ge 0$, and $g(x,y)+x\frac{\partial g(x,y)}{\partial x} \le 1$ and $g(x,y)+y\frac{\partial g(x,y)}{\partial y} \le 1$.

The marginal CDF and pdf of a Weibull r.v. U are $F_U(u)=(1-e^{-u^\gamma})I_{(0,\infty)}(u)$ and $f_U(u)=\gamma u^{\gamma-1}e^{-u^\gamma}I_{(0,\infty)}(u)$, respectively, where $I_{(0,\infty)}(u)$ is an indicator function and $0<\gamma$. The marginal CDF and pdf of a second Weibull r.v., V, are identical except for replacement of the parameter γ by γ'. The g(x,y) for this distribution is

$$g(x,y) = cx^m(1-x)^a y^n(1-y)^b \qquad (2.10)$$

where m and n are nonnegative integers, $0<a$, $0<b$, and c satisfies the restrictions involving g(x,y). Thus, a,b,c,m and n are dependence parameters. The bivariate CDF and pdf of Weibull r.v.'s U and V is

$$F_{U,V}(u,v) = xy-c\sum_{j=0}^{m+1}\sum_{k=0}^{n+1}\binom{m+1}{j}\binom{n+1}{k}(-1)^{j+k}(1-x)^{a+j}(1-y)^{b+k} \qquad (2.11)$$

and

$$f_{U,V}(u,v) = \{1-c\sum_{j=0}^{m+1}\sum_{k=0}^{n+1}\binom{m+1}{j}\binom{n+1}{k}(a+j)(b+k) \cdot(-1)^{j+k}(1-x)^{a+j-1}(1-y)^{b+k-1}\}f_U(u)f_V(v), \qquad (2.12)$$

respectively. The moments of r.v.'s U and V are given by $E(U^sV^t) = \Gamma[(s/\gamma)+1]\Gamma[(t/\gamma')+1]$

$$\cdot\{1-c\sum_{j=0}^{m+1}\sum_{k=0}^{n+1}\frac{\binom{m+1}{j}\binom{n+1}{k}(-1)^{j+k}}{(a+j)^{(s/\gamma)+1}(b+k)^{(t/\gamma')+1}}\}.$$

Let $R=U/V$. If U and V are bivariate Weibull r.v.'s with $\gamma=\gamma'$, then (2.12) yields the pdf of R given by

$$f_R(r)=\gamma r^{\gamma-1}[(r^\gamma+1)^{-2}-c\sum_{j=0}^{m+1}\sum_{k=0}^{n+1}\binom{m+1}{j}\binom{n+1}{k}(a+j)(b+k) \cdot(-1)^{j+k}\{(a+j)r^\gamma+(b+k)\}^{-2}],\ 0 < r. \qquad (2.13)$$

The moments of R follow from the identity $E(R^s)=E(U^sV^{-s})$.

E. Beta-P

The marginal CDF and pdf of a beta-P r.v. U are $F_U(u)=\{1-(1+u^\theta)^{-\alpha}\}I_{(0,\infty)}(u)$ and $f_U(u)=\alpha\theta u^{\theta-1}(1+u^\theta)^{-(\alpha+1)}I_{(0,\infty)}(u)$, respectively, where $0<\alpha$ and $0<\theta$. The marginal CDF and pdf of a second beta-P r.v. V are identical except for replacement of the parameter set (α,θ) by (α',θ'). Using the g(x,y) chosen for the Weibull (2.10), the bivariate CDF and pdf of beta-P r.v.'s U and V are again (2.11) and (2.12). The moments of r.v.'s U and V are given by

$$E(U^sV^t)=\alpha\alpha'[B(1+\frac{s}{\theta},\alpha-\frac{s}{\theta})B(1+\frac{t}{\theta'},\alpha'-\frac{t}{\theta'})$$
$$-c\sum_{j=0}^{m+1}\sum_{k=0}^{n+1}\binom{m+1}{j}\binom{n+1}{k}(a+j)(b+k)(-1)^{j+k}$$
$$\cdot B\{1+\frac{s}{\theta},\alpha(a+j)-\frac{s}{\theta}\}B\{1+\frac{t}{\theta'},\alpha'(b+k)-\frac{t}{\theta'}\}].$$

Again, let R=U/V. If U and V are bivariate beta-P r.v.'s with $\theta=\theta'$, then (2.12) again yields the pdf of R given by

$$f_R(r)=\alpha\alpha'\theta r^{\theta-1}[B(2,\alpha+\alpha')F(\alpha+1,2;\alpha+\alpha'+2;1-r^\theta)$$
$$-c\sum_{j=0}^{m+1}\sum_{k=0}^{n+1}\binom{m+1}{j}\binom{n+1}{k}(a+j)(b+k)(-1)^{j+k}B\{2,\alpha(a+j)$$
$$+\alpha'(b+k)\}F\{\alpha(a+j)+1,2;\alpha(a+j)+\alpha'(b+k)+2;1-r^\theta\}],$$
$$0 < r. \quad (2.14)$$

Again the moments of R follow from the identity $E(R^s)=E(U^sV^{-s})$.

F. Beta-κ

The marginal CDF and pdf of a beta-κ r.v. U are $F_U(u)=\{1-(1+u^\theta)^{-1}\}^\alpha I_{(0,\infty)}(u)$ and $f_U(u)=\alpha\theta u^{\alpha\theta-1}(1+u^\theta)^{-(\alpha+1)}I_{(0,\infty)}(u)$, respectively, where $0<\alpha$ and $0<\theta$. The marginal CDF and pdf of a second beta-κ r.v., V, are identical except for replacement of the parameter set (α,θ) by (α',θ'). A convenient choice of g(x,y) for this distribution is $g(x,y)=cx^a(1-x)^my^b(1-y)^n$ where m and n are positive integers, $0\le a$, $0\le b$, and c satisfies the restrictions involving g(x,y). Again, a,b, c,m and n are dependence parameters. The bivariate CDF and pdf of beta-κ r.v.'s U and V are

$$F_{U,V}(u,v) = xy$$
$$-c\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}(-1)^{j+k}x^{a+j+1}y^{b+k+1} \quad (2.15)$$

and

$$f_{U,V}(u,v) = \{1-c\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}$$
$$(a+j+1(b+k+1)(-1)^{j+k}x^{a+j}y^{b+k}\}f_U(u)f_V(v), \quad (2.16)$$

respectively. The moments of r.v.'s U and V are given by

$$E(U^sV^t) = \alpha\alpha'[(\alpha+\frac{s}{\theta},1-\frac{s}{\theta})B(\alpha'+\frac{t}{\theta'},1-\frac{t}{\theta'})$$
$$-c\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}(a+j+1)(b+k+1)(-1)^{j+k}$$
$$\cdot B\{\alpha(a+j+1)+\frac{s}{\theta},1-\frac{s}{\theta}\}B\{\alpha'(b+k+1)+\frac{t}{\theta'},1-\frac{t}{\theta'}\}].$$

Let R=U/V. If U and V are bivariate beta-κ r.v.'s with $\theta=\theta'$, then (2.16) yields the pdf of R given by

$$f_R(r)=\alpha\alpha'\theta r^{\alpha\theta-1}[B(\alpha+\alpha',2)F(\alpha+1,\alpha+\alpha';\alpha+\alpha'+2;1-r^\theta)$$
$$-c\sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k}(a+j+1)(b+k+1)(-1)^{j+k}r^{\alpha\theta(a+j)}$$
$$\cdot B\{\alpha(a+j+1)+\alpha'(b+k+1),2\}F\{\alpha(a+j+1)+1,\alpha(a+j+1)$$
$$+\alpha'(b+k+1);\alpha(a+j+1)+\alpha'(b+k+1)+2;1-r^\theta\}],$$
$$0 < r, \quad (2.17)$$

and the moments of R follow from the identity $E(R^s) = E(U^sV^{-s})$.

III. CONCLUDING COMMENTS

This paper has reviewed a number of known results and presented some new results for the distributions of ratios of r.v.'s. The new results have (i) given extensions and simplifications of previous work involving the ratio of correlated gamma r.v.'s, (ii) extended Malik's work on the ratio of independent generalized gamma r.v.'s to the correlated case (iii) extended Reitz's and Marsaglia's results to the general case of the ratio of correlated beta r.v.'s of the first kind, (iv) extended Aitchison and Brown's work on the ratio of independent log-normal r.v.'s to the correlated case, and (v) presented additional results for ratios of dependent Weibull, beta-P, and beta-κ r.v.'s. In conjunction with this work, a number of new selected bivariate distributions are also presented.

In applications of specific ratio distributions [e.g., gamma (see Flueck and Holland, 1976) and log-normal (see Aitchison and Brown, 1966)], standard estimators will be appropriate. For the other ratio distributions, additional estimation procedures should be advantageous. Also, in some applications the use of bivariate distributions permitting different scale parameters (e.g., log-normal, Weibull, beta-P, and beta-κ) should allow closer modeling of the data.

REFERENCES

Aitchison, J. and Brown, J. A. C. (1966). The Lognormal Distribution. Cambridge: Cambridge University Press.

Block, H. W. and Rao, B. R. (1973). A beta warning-time distribution and a distended beta distribution. Sankhyā B 34, 79-84.

Bose, S. (1935). On the distribution of the ratio of variances of two samples drawn from a given normal bivariate correlated population. Sankhyā 2, 65-72.

Broadbent, S. R. (1954). The quotient of a rectangular or triangular and a general variate. Biometrika 41, 330-7.

Cherian, K. C. (1941). A bivariate correlated gamma-type distribution function. J. Indian Math. Soc. 5, 133-44.

Cochran, W. G. (1963). Sampling Techniques, 2nd edition. New York: Wiley.

Craig, C. C. (1929). The frequency function of y/x. Ann. of Math. 30, 471-86.

David, F. N. and Fix, E. (1961). Rank correlation and regression in a nonnormal surface. Proc. 4th Berkeley Symp. 1, 177-97.

Farlie, D. S. G. (1960). The performance of some correlation coefficients for a general bivariate distribution. Biometrika 47, 307-23.

Fieller, E. C. (1932). The distribution of the index in a normal bivariate population. Biometrika 24, 428-40.

Fisher, R. A. (1924). On a distribution yielding the error functions of several well known statistics. Proceedings of the International Mathematical Congress, Toronto, 2, 805-13.

Flueck, J. A. and Holland, B. (1974). The moments and distribution of the ratio of two gamma variables. 1973 Proceedings, Social Statistics Section, American Statistical Association, 252-5.

Flueck, J. A., Holland, B. (1976). Ratio estimators and some inherent problems in their utilization. J. Appl. Meteor., 15, 535-43.

Flueck, J. A., Holland, B., and Lee, R. (1975). Distribution of the ratio of correlated sums of gamma variables. 1974 Proceedings, Social Statistics Section, American Statistical Association, 285-91.

Geary, R. C. (1930). The frequency distribution of the quotient of two normal variates. J. R. Statist. Soc. 93, 442-6.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). Table of Integrals, Series, and Products. New York: Academic Press.

Greenwood, M. and White, J. D. C. (1910). A biometric study of phagocytosis with special reference to the opsonic index. Biometrika 7, 505-30.

Huntington, E. V. (1939). Frequency distributions of products and quotients. Ann. Math. Statist. 10, 195-8.

Krishnaiah, P. R., Hagis, P. and Steinberg, L. (1963). A note on the bivariate chi distribution. SIAM Rev. 5, 140-4.

Kullback, S. (1936). The distribution laws of the differences and quotient of variables independently distributed in Pearson type III laws. Ann. Math. Statist. 7, 51-3.

Malik, H. J. (1967). Exact distribution of the quotient of independent generalized gamma variables. Canad. Math. Bull. 10, 463-5.

Marsaglia, G. (1965). Ratio of normal variables and ratios of sums of uniform variables. J. Amer. Statist. Assoc. 60, 193-204.

Merrill, A. S. (1928). Frequency distribution of an index when both the components follow the normal law. Biometrika 20A, 53-63.

Mielke, P. W. (1975). Conveient beta distribution likelihood techniques for describing and comparing meteorological data. J. Appl. Meteor. 14, 985-90.

Mielke, P. W. and Johnson, E. S. (1974). Some generalized beta distributions of the second kind having desirable application features in hydrology and meteorology. Water Resour. Res. 10, 223-6. Corrections 11, 606; 12, 827.

Pearson, K. (1910). On the constants of index distributions as deduced from the like constants for the components of the ratio, with special reference to the opsonic index. Biometrika 7, 531-41.

Press, S. J. (1969). The t-ratio distribution, J. Amer. Statist. Assoc. 64, 242-252.

Rao, C. R. (1952). Advanced Statistical Methods in Biometric Research. New York: Wiley.

Rao, B. R. and Garg, M. L. (1969). A note on the generalized (positive) Cauchy distribution. Canad. Math. Bull. 12, 865-8.

Rietz, H. L. (1936). On the frequency distribution of certain ratios. Ann. Math. Statist. 7, 145-53.

Stacy, E. W. (1962). A generalization of the gamma distribution. Ann. Math. Statist. 33, 1187-92.