JACKNIFE VARIANCE ESTIMATION FOR NEAREST NEIGHBOR IMPUTATION

Jiahua Chen, University of Waterloo
Jun Shao, University of Wisconsin-Madison
Jun Shao, Department of Statistics, 1210 Dayton Street, Madison WI 53706 U.S.A.

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Abstract:
Nearest neighbor imputation is a popular hot deck imputation method used to compensate for nonresponse in sample surveys. Although this method has a long history of application, the problem of variance estimation after nearest neighbor imputation has not been fully investigated. Since nearest neighbor imputation is a nonparametric method, a nonparametric variance estimation technique such as the jackknife is desired. We show that the naive jackknife that treats imputed values as observed data produces serious underestimation. We also show that Rao and Shao's (1992) adjusted jackknife or the jackknife with each pseudoreplicate re-imputed, which produces asymptotically unbiased and consistent jackknife variance estimators for other imputation methods (such as mean imputation, random hot deck imputation, ratio or regression imputation), produces serious overestimation in the case of nearest neighbor imputation. Two partially re-imputed and a partially adjusted jackknife variance estimators are proposed in this article and shown to be asymptotically unbiased and consistent. Some simulation results are provided to examine finite sample properties of these jackknife variance estimators.

1 Introduction

Imputation is commonly applied to compensate for item nonresponse in sample surveys (Kalton and Kasprzyk 1986, Sedransk 1985, Rubin 1987). The nearest neighbor imputation (NNI) method is used in many survey agencies such as the U.S. Bureau of Labor Statistics, the U.S. Census Bureau, and Statistics Canada. Consider a bivariate sample \((x_1, y_1), \ldots, (x_n, y_n)\) and suppose that \(y_1, \ldots, y_r\) are observed (respondents), \(y_{r+1}, \ldots, y_n\) are missing (nonrespondents), and all \(x\)-values are observed. The NNI method imputes a missing \(y_i\) by \(y_j\), where \(1 \leq i \leq r\) and \(i\) is the nearest neighbor of \(j\) measured by the \(x\)-variable, i.e., \(i\) satisfies

\[
\left| x_i - x_j \right| = \min_{1 \leq l \leq r} \left| x_i - x_l \right|.
\]

If there are tied \(x\)-values, then there may be multiple nearest neighbors of \(j\) and \(i\) is randomly selected from them.

The NNI method has some nice features. First, it is a hot deck method in the sense that nonrespondents are substituted by respondents from the same variable; Second, it is shown in Chen and Shao (1999) that the NNI method provides asymptotically unbiased and consistent estimators for population means as well as quantiles. Third, the NNI method may be more efficient than other hot deck methods that do not make use of auxiliary information provided by the \(x\)-values (e.g., the mean imputation method). Finally, the NNI method uses a nonparametric model relating \(y\) and \(x\) (see Section 2) and, hence, it is expected to be more robust against model violations than methods based on parametric models, such as ratio imputation and regression imputation.

In this article we focus on jackknife variance estimation for the sample means (or weighted averages) based on data imputed by NNI. It is known that naive jackknife variance estimator often underestimates variance in the presence of imputation and some adjustments are necessary (Rao and Shao, 1992). Full implementation of the adjustment principle, however, overestimates the variance of the sample mean based on NNI. In view of these, we propose a partially re-imputed jackknife and a partially adjusted jackknife, and show that they produce asymptotically unbiased and consistent variance estimators for the sample means based on NNI, under some weak conditions for stratified samples.

The rest of the article is organized as follows. Some notation, assumptions, and details for NNI and the jackknife are given in Section 2. The asymptotic biases of the naive jackknife variance estimator
and Rao and Shao's adjusted jackknife variance estimator are derived in Section 3. Partially re-imagined and partially adjusted jackknife variance estimators are proposed in Section 4 and they are shown to be asymptotically unbiased and consistent. Section 5 contains some simulation results for these jackknife variance estimators, using a population that is close to a real data set from 1988 Current Population Survey (Valliant 1993).

2 Preliminaries

2.1 Sampling Design and Model

Let \( \mathcal{P} \) be a finite population containing indices 1, \ldots, \( N \). Assume that \( \mathcal{P} \) is stratified into \( H \) strata with \( N_h \) units in the \( h \)-th stratum and that \( n_h \geq 2 \) units are selected without replacement from stratum \( h \) according to some probability sampling plan, independently across the strata. The overall sampling fraction \( n/N \) is assumed negligible, where \( n = \sum_h n_h \), although \( n_h/N_h \) may be non-negligible for some \( h \). Let \( S \) denote the sample. According to the sampling plan, survey weights \( w_i, i \in S \), are constructed so that for any set of values \( \{ z_i : i \in \mathcal{P} \} \),

\[
E_{S} \left( \sum_{i \in S} w_i z_i \right) = \frac{1}{N} \sum_{i=1}^{N} z_i,
\]

where \( E_{S} \) is the expectation with respect to \( S \). This sampling design is commonly used in many business surveys conducted at the U.S. Bureau of Labor Statistics and the U.S. Census Bureau.

Let \( y \) be a variable of interest and \( x \) be an auxiliary variable. Let \( a \) be the response indicator for \( y \) (i.e., for the \( i \)-th unit, \( a_i = 1 \) if \( y_i \) is a respondent and \( a_i = 0 \) otherwise). The validity of NNI is based on the following model assumption.

**Assumption A.** The finite population \( \mathcal{P} \) is divided into \( K \) imputation classes such that within each imputation class, \( (x_i, y_i, a_i) \)’s are iid from a superpopulation and \( P(a_i = 1|x_i, y_i) = P(a_i = 1|x_i) \). \( (x_i, y_i, a_i) \)’s from different imputation classes are independent. NNI is carried out within each imputation class.

We assume that \( K \) is fixed and the number of units in each imputation class is large. This is necessary for the validity of NNI, in fact, for any model-based and nonparametric imputation method (Val- liant 1993).

2.2 Nearest Neighbor Imputation

Let \( S_k \) be the set of indices of sampled units in imputation class \( k \), \( R_k \) be the set of indices of \( y \)-respondents in imputation class \( k \), and \( N_k \) be the set of indices of \( y \)-nonrespondents in imputation class \( k \) (\( S_k = R_k \cup N_k \), \( k = 1, \ldots, K \)). Under assumption A, conditional on \( r_k \), the number of respondents in \( S_k \), \( \{(y_i, x_i), i \in R_k\} \) and \( \{(y_i, x_i), i \in N_k\} \) are independent sets of iid random vectors from two possibly different distributions.

For \( j \in N_k \), let \( \hat{y}_j = y_i \) denote the value imputed by NNI, where \( i \) is selected according to

\[
|x_i - x_j| = \min_{i \in R_k} |x_i - x_j|. \tag{2}
\]

Note that (1) is a special case of (2). We focus on the case where the distribution of \( x \) related to the superpopulation is continuous so that there are no tied \( x \)-values. The NNI sample mean can be written as

\[
\bar{y}_{NNI} = \sum_{k=1}^{K} \sum_{i \in R_k} \left( 1 + d^{(k)}_i \right) w_i y_i, \tag{3}
\]

where

\[
d_i = \begin{cases} 1 & \text{if } i \text{ is the nearest neighbor of } j \\ 0 & \text{otherwise} \end{cases}
\]

and \( d^{(k)}_i = \sum_{j \in N_k} w_i^{-1} w_j d_{ij} \).

Chen and Shao (1999) showed that \( \bar{y}_{NNI} \) in (3) is asymptotically unbiased (for the population mean \( \bar{Y} = N^{-1} \sum_{i=1}^{N} y_i \)) and has the following asymptotic variance

\[
V = \sum_{k=1}^{K} E \left[ \sum_{i \in R_k} \left( 1 + d^{(k)}_i \right)^2 w_i^2 \text{Var}(y_i | x_i) \right] + \sum_{k=1}^{K} \text{Var} \left[ \sum_{i \in S_k} w_i \psi_k(x_i) \right], \tag{4}
\]

where \( \psi_k(x) = E(y|x) \) in the \( k \)-th imputation class. Their results were established under the stratified sampling design described in Section 2.1, assumption A, and the following technical assumption.

**Assumption B.** (i) The total number of sampled units \( n \to \infty \) and \( m_k^{-1} = O(n^{-1}) \), \( k = 1, \ldots, K \), where \( m_k \) is the number of sampled units in imputation class \( k \).

(ii) The survey weights satisfy \( \max_i w_i = O(n^{-1}) \).

(iii) There exist constants \( -\infty < M_1 < M_2 \leq \infty \) and \( C \) such that the function \( \psi_k(x) \) is monotone when \( x < M_1 \) or \( x > M_2 \), and \( |\psi_k(t) - \psi_k(s)| \leq C |t - s| \) when \( M_1 \leq s, t \leq M_2 \).

(iv) The marginal distribution of \( x \) has a density, \( E|x|^3 < \infty \), \( E|x|^6 < \infty \), and \( E|y_i|^6 < \infty \).

(v) The response probability \( P(a_i = 1|x) \) satisfies \( \in R_D P(a_i = 1|x) > 0 \), where \( D \) is the support of the marginal distribution of \( x \).
2.3 The Jackknife and the Adjusted Jackknife

If \( \psi_k(x) \) and \( \text{Var}(y|x) \) in (4) have parametric forms, then we can estimate the variance in (4) by substitution (Chen and Shao 1999). Since NNI is non-parametric (i.e., the function forms of \( \psi_k(x) \) and \( \text{Var}(y|x) \) are not known), a nonparametric variance estimation method is preferred.

The simplest nonparametric resampling method for variance estimation is the jackknife method, which estimates the variance of an estimator \( \hat{\theta} \) by

\[
V_{\text{Jack}} = \frac{1}{m_k - 1} \sum_{j \in S_k} \left( \hat{\theta}(j) - \hat{\theta} \right)^2,
\]

where \( m_k \) is the number of units in \( S_k \) and \( \hat{\theta}(j) \) is the same as \( \hat{\theta} \) but is based on the jth jackknife pseudoreplicate, which is the dataset obtained by changing survey weight \( w_i \) to \( w_i / m_k - 1 \) if \( i \in j \in S_k, \) \( i \neq j \) \( w_i \) if \( i = j. \)

Note that we adjust survey weights using \( m_k \)'s, which are the imputation class sizes (or poststratum sizes), not the original stratum sizes \( n_k \)'s. This is justified by the superpopulation model assumption in assumption A. In the case of nonresponse, the jackknife estimator \( V_{\text{Jack}} \) is asymptotically unbiased and consistent when \( \hat{\theta} \) is a differentiable function of sample means.

In the case of \( \hat{\theta} = \bar{y}_{\text{NNI}}, \) however, directly applying formula (5) and treating imputed nonrespondents as observed data produces a \( V_{\text{Jack}} \) that underestimates the asymptotic variance of \( \bar{y}_{\text{NNI}}. \) The precise form of the bias of \( V_{\text{Jack}} \) under NNI is given in Section 3.

For mean imputation and random hot deck imputation (which imputes \( y \)-nonrespondents in \( S_k \) by a random sample from \( y \)-respondents in \( S_k \), Rao and Shao (1992) proposed to first adjust the imputed values in each jackknife pseudoreplicate and then apply formula (5) with \( \hat{\theta}(j) \) computed based on the adjusted jth pseudoreplicate. Rao and Shao's adjustment can be described as follows. In the jth pseudoreplicate, each imputed value \( \tilde{y}_i \) is adjusted to

\[
\tilde{y}_{i,j}^{(k)} = \tilde{y}_i + \bar{E}(\tilde{y}_{i,j}) - \bar{E}(\tilde{y}),
\]

where \( \tilde{y}_{i,j} \) is the imputed value for \( y_i \) using the respondents in the jth pseudoreplicate and \( \bar{E} \) is the expectation for random imputation. For any non-random imputation (given the observed data), adjustment (6) reduces to

\[
\tilde{y}_{i,j}^{(k)} = \tilde{y}_{i,j},
\]

i.e., Rao and Shao's adjustment amounts to re-imputing nonrespondents in each pseudoreplicate using the observed data within the same pseudoreplicate.

Let \( \tilde{y}_{i,j}^{(k)} \) be the same as \( \tilde{y}_{i,j}^{(k)} \) but based on the jth jackknife pseudoreplicate re-imputed according to (7) or, equivalently, adjusted according to (6). Then the re-imputed or adjusted jackknife variance estimator is

\[
V_{\text{Jack}}^{a} = \frac{1}{m_k - 1} \sum_{j \in S_k} \left( \tilde{y}_{i,j}^{(k)} - \tilde{y} \right)^2.
\]

Let us focus on imputation class k. Recall that \( R_k \) is the set of respondents and \( N_k \) is the set of non-respondents. Consider the jth pseudoreplicate with a respondent \( y_j \) in \( R_k \). Let \( y_j \) be a non-responder in \( N_k \). If the original NNI value \( y_j \) is not \( y_j \), then the re-imputed value \( \tilde{y}_{i,j} \) is still \( y_j \). If \( y_j = y_j \), i.e., \( j \) is the nearest neighbor of \( t \), then the re-imputed value \( \tilde{y}_{i,j} \) is the \( y \)-value of the second nearest neighbor of \( t \). Let \( x_{j_1} \in R_k \) and \( x_{j_2} \in R_k \) be two nearest neighbor values of \( x_j \) (\( x_{j_1} < x_j < x_{j_2} \) if \( x_j \) is not the smallest or largest \( x \)-value in \( R_k \); otherwise \( x_{j_1} = x_{j_2} \)). Then the second nearest neighbor of \( t \) must be either \( j_1 \) or \( j_2 \), i.e., \( \tilde{y}_{i,j} = y_{j_1} \) or \( y_{j_2} \). Let

\[
c_j = \begin{cases} 
1 & j_1 \text{ is the second nearest neighbor of } t \\
0 & j_2 \text{ is the second nearest neighbor of } t
\end{cases}
\]

given that \( j \) is the nearest neighbor of \( t \),

\[
c_j = \sum_{t \in N_k} \frac{w_t}{w_j} c_j,
\]

and let \( d_j^{(k)} \) be the same as that in (3). Then

\[
(m_k - 1)(\bar{y}_{\text{NNI}} - \tilde{y}_{i,j})
= m_k w_j y_j - \tilde{y}_k
+ m_k w_j [d_j^{(k)} y_j - c_j^{(k)} y_{j_1} - (d_j^{(k)} - c_j^{(k)}) y_{j_2}],
\]
if $y_j$ is a respondent in $\mathcal{R}_k$, and
\begin{equation}
(m_k - 1)(\bar{y}_{NN1} - \bar{y}_{n1}^{(j)}) = m_k w_j \bar{y}_j - \bar{y}_k, \tag{10}
\end{equation}
if $y_j$ is a nonrespondent in $\mathcal{N}_k$, where $\bar{y}_k = \sum_{i \in \mathcal{R}_k} (1 + d_j^{(k)}) w_i y_i$.

For convenience, write $e_i = y_i - \psi_k(x_i)$. Then $e_i$'s are uncorrelated, conditional on $x_i$'s, and $E(e_i|x_i) = 0$ and $\text{Var}(e_i|x_i) = \text{Var}(y_i|x_i)$. It follows from (9) that when $y_j$ is a respondent in imputation class $k$,
\begin{equation}
(m_k - 1)(\bar{y}_{NN1} - \bar{y}_{n1}^{(j)}) = m_k w_j [(1 + d_j^{(k)}) e_j - c_j \epsilon_j - (d_j^{(k)} - c_j^{(k)}) \epsilon_j] + m_k w_j z_j + \eta_j^{(k)},
\end{equation}
where $z_j = \psi_k(x_j) - \mu_k$, $\mu_k = E[\psi_k(x_i)]$, 
\begin{equation}
\eta_j^{(k)} = -m_k w_j [c_j^{(k)} (z_j - z_j) + (d_j^{(k)} - c_j^{(k)}) (z_j - z_j)] + m_k w_j \psi_k - \psi_k(x_j) \end{equation}
and $\psi_k (or \bar{e}_k)$ is the same as $\bar{y}_k$ but with $y_i$ replaced by $\psi_k(x_i)$ (or $e_i$).

It follows from Lemma 1 in the Appendix that
\begin{equation}
E \left[ \sum_{j \in \mathcal{R}_k} \left( \frac{\eta_j^{(k)}}{m_k - 1} \right)^2 \right] = o(n^{-1}), \tag{11}
\end{equation}
where $E$ is the joint expectation with respect to model and sampling. Let $\mu_j^{(k)} = (1 + d_j^{(k)})^2 + (c_j^{(k)})^2 + (d_j^{(k)} - c_j^{(k)})^2 w_j$. Using the fact that $e_j$'s are conditionally uncorrelated and assuming that the function $\sigma^2(x) = \text{Var}(y|x)$ is continuous in $x$, we obtain that
\begin{equation}
E \left[ \sum_{j \in \mathcal{R}_k} \left( \bar{y}_{NN1} - \bar{y}_{n1}^{(j)} \right)^2 \right] = E \left[ \sum_{j \in \mathcal{R}_k} \mu_j^{(k)} \text{Var}(y_j|x_j) \right] + E \left( \sum_{j \in \mathcal{R}_k} w_j^2 z_j^2 \right) + o(n^{-1}).
\end{equation}
For nonrespondents in $\mathcal{N}_k$, a similar argument shows that
\begin{equation}
E \left[ \sum_{j \in \mathcal{N}_k} \left( \bar{y}_{NN1} - \bar{y}_{n1}^{(j)} \right)^2 \right] = E \left[ \sum_{j \in \mathcal{N}_k} d_j^{(k)} w_j^2 \text{Var}(y_j|x_i) \right] + E \left( \sum_{j \in \mathcal{N}_k} w_j^2 z_j^2 \right) + o(n^{-1}),
\end{equation}
where the last equality follows from Lemma 2 in the Appendix.

The naive jackknife variance estimator is defined by (5) with $\hat{\theta} = \bar{y}_{NN1}$ and treating imputed values as observed data. By omitting the last term on the right hand side of (9) and using a similar argument, we obtain that
\begin{equation}
E \left[ \sum_{j \in \mathcal{R}_k} \left( \bar{y}_{NN1} - \bar{y}_{n1}^{(j)} \right)^2 \right] = E \left[ \sum_{j \in \mathcal{R}_k} w_j^2 \text{Var}(y_j|x_j) \right] + E \left( \sum_{j \in \mathcal{R}_k} w_j^2 z_j^2 \right) + o(n^{-1})
\end{equation}
and
\begin{equation}
E \left[ \sum_{j \in \mathcal{N}_k} \left( \bar{y}_{NN1} - \bar{y}_{n1}^{(j)} \right)^2 \right] = E \left[ \sum_{j \in \mathcal{N}_k} d_j^{(k)} w_j^2 \text{Var}(y_j|x_j) \right] + E \left( \sum_{j \in \mathcal{N}_k} w_j^2 z_j^2 \right) + o(n^{-1}).
\end{equation}

Combining these results and imputation classes, we obtain the following result on the biases of the naive jackknife variance estimator $\hat{v}_{JACK}$ and the re-imputed (or adjusted) jackknife variance estimator $\hat{v}_{RJACK}$.

**Theorem 1.** Suppose that assumptions A and B hold and that the function $\sigma^2(x) = \text{Var}(y|x)$ is continuous in $x$. Then
\begin{equation}
E(\hat{v}_{JACK}) = \sum_{k=1}^{\mathcal{K}} E \left[ \sum_{i \in \mathcal{R}_k} (1 + d_j^{(k)}) w_i^2 \text{Var}(y_i|x_i) \right] + \sum_{k=1}^{\mathcal{K}} \text{Var} \left[ \sum_{i \in \mathcal{S}_k} w_i \psi_k(x_i) \right] + o(n^{-1})
\end{equation}
and
\begin{equation}
E(\hat{v}_{RJACK}) = \sum_{k=1}^{\mathcal{K}} E \left[ \sum_{i \in \mathcal{R}_k} (1 + d_j^{(k)}) w_i^2 \text{Var}(y_i|x_i) \right] + \sum_{k=1}^{\mathcal{K}} \text{Var} \left[ \sum_{i \in \mathcal{S}_k} w_i \psi_k(x_i) \right] + o(n^{-1}).
\end{equation}

Compared with result (4), we conclude that the naive jackknife variance estimator $\hat{v}_{JACK}$ has a negative bias and the re-imputed jackknife variance estimator has a positive bias. Some numerical examples of these biases are given in the simulation study in Section 5.
4 Partially Re-imputed or Partially Adjusted Jackknife

In view of the fact that the naive jackknife underestimates and the re-imputed jackknife (which is the same as the adjusted jackknife) overestimates, we propose some jackknife methods with partial re-imputation or partial adjustment. The first partially re-imputed jackknife can be described as follows. For \( j \in T g_k \), nonrespondents in pseudoreplicate \( j \) are re-imputed with probability \( p_{Jk} \) and not re-imputed with probability \( 1 - p_{Jk} \). Let \( \delta \) be the indicator that equals 1 when re-imputation is conducted. Then a partially re-imputed jackknife variance estimator is given by

\[
\hat{\text{Var}}(y_j | x_j) = \sum_{k=1}^{K} \frac{m_k - 1}{m_k} \sum_{j \in S_k} \left[ \delta_j \left( \bar{y}_{j,(k)} - \bar{y}_{NNI} \right)^2 + (1 - \delta_j) \left( \bar{y}_{j,(k)} - \bar{y}_{NNI} \right)^2 \right].
\]

(12)

(Recall that for pseudoreplicate \( j \) with \( j \in N_k \), \( \bar{y}_{j,(k)} = \bar{y}_{j,(NNI)} \).) Using the same argument in establishing Theorem 1, we obtain that

\[
E(\hat{\text{Var}}_{PRI}(y_j | x_j)) = \sum_{k=1}^{K} \sum_{j \in R_k} \left( \rho_{J(k)}^{(k)} \right) \text{Var}(y_j | x_j) + o(n^{-1}).
\]

By choosing \( \rho_{J(k)}^{(k)} \) so that the coefficient in front of \( \text{Var}(y_j | x_j) \) equals \((1 + d_{J(k)}^{(k)})^2 \), we can obtain an asymptotically unbiased \( \hat{\text{Var}}_{PRI}^{(k)} \). This leads to

\[
\rho_{J(k)}^{(k)} = \frac{d_{J(k)}^{(k)} (1 + d_{J(k)}^{(k)})}{(1 + d_{J(k)}^{(k)})^2 + (c_{J(k)}^{(k)})^2 + (d_{J(k)}^{(k)} - c_{J(k)}^{(k)})^2 - 1}
\]

(\( \rho_{J(k)}^{(k)} = 0 \) if \( d_{J(k)}^{(k)} = 0 \)).

To compute \( \hat{\text{Var}}_{PRI}^{(k)} \) in (12), both \( d_{J(k)}^{(k)} \) and \( c_{J(k)}^{(k)} \) have to be computed in order to obtain the probability \( \rho_{J(k)}^{(k)} \). Note that the computation of \( c_{J(k)}^{(k)} \) is more complicated than that of \( d_{J(k)}^{(k)} \), since second nearest neighbors have to be located. The following slight modification of the previously described procedure avoids the computation of \( c_{J(k)}^{(k)} \). Instead of re-imputing nonrespondents in pseudoreplicate \( j \) by either \( y_{j1} \) or \( y_{j2} \), where \( j1 \) and \( j2 \) are defined in (8), we re-impute nonrespondents by the average \((y_{j1} + y_{j2})/2\), whenever re-imputation is needed.

The partially re-imputed jackknife variance estimator \( \hat{\text{Var}}_{PRI}^{(k)} \) with this modification in the re-imputation procedure is then asymptotically unbiased if \( \rho_{J(k)}^{(k)} = 2(1 + d_{J(k)}^{(k)}) / (3d_{J(k)}^{(k)} + 4) \) \( (\rho_{J(k)}^{(k)} = 0 \) if \( d_{J(k)}^{(k)} = 0) \).

Instead of randomizing re-imputation, we can use a partial adjustment, i.e., modify adjustment (7) to

\[
\bar{y}_{i,j}^{(k)} = \bar{y}_i + g_{j}^{(k)} (\bar{y}_{i,j} - \bar{y}_i)
\]

(13)

with a constant \( g_{j}^{(k)} \in [0, 1] \) to be specified later. If \( g_{j}^{(k)} = 1 \), then adjustment (13) is the same as adjustment (7); if \( g_{j}^{(k)} = 0 \), then there is no adjustment; if \( 0 < g_{j}^{(k)} < 1 \), then there is a partial adjustment. For simplicity, we may use \( \bar{y}_{i,j} = (y_{j1} + y_{j2})/2 \) when \( y_j \) is a respondent.

Our third proposed jackknife variance estimator is obtained by applying formula (5) with \( \bar{y}^{(j)} \) being the sample mean computed based on the \( j \)th pseudoreplicate adjusted according to (13). This estimator is called the partially adjusted jackknife variance estimator and denoted by \( \hat{\text{Var}}_{PA}^{(k)} \). Note that although re-imputed jackknife and adjusted jackknife are the same, partially re-imputed jackknife and partially adjusted jackknife are generally different.

Using the same argument in establishing Theorem 1, we can also show that \( \hat{\text{Var}}_{PRI}^{(k)} \) is asymptotically unbiased when

\[
g_{j}^{(k)} = \sqrt{\frac{6(d_{J(k)}^{(k)})^2 + 6d_{J(k)}^{(k)} + 4 - 2}{3d_{J(k)}^{(k)}}}
\]

(\( g_{j}^{(k)} = 0 \) if \( d_{J(k)}^{(k)} = 0 \)). It is easy to verify that \( 0 \leq g_{j}^{(k)} < 1 \).

The following result summarizes the asymptotic performance of three proposed jackknife variance estimators.

**Theorem 2.** Assume the conditions in Theorem 1. Assume further that \( nV \) is bounded away from 0, where \( V \) is the asymptotic variance of \( \bar{y}_{NNI} \) given by (4). Then

\[
\frac{E(\hat{\text{Var}}_{PRI}^{(k)} / V)}{V} \rightarrow 1
\]

(14)

and

\[
\frac{\hat{\text{Var}}_{PRI}^{(k)} / V}{V} \rightarrow 1 \text{ in probability. (15)}
\]

Results (14) and (15) also hold with \( \hat{\text{Var}}_{PRI}^{(k)} \) replaced by \( \hat{\text{Var}}_{PA}^{(k)} \) or \( \hat{\text{Var}}_{JACK}^{(k)} \).

The proof is given in the Appendix.

5 Simulation Results

As a complement to our theory, we present in this section some results from a simulation study. We
examine five jackknife variance estimators for $\hat{y}_{NI}$: the naive jackknife estimator $v_{JACK}$, the re-imputed jackknife estimator $v_{JACK}^R$, two partially re-imputed jackknife estimators $v_{PRI}^R$ and $v_{PRI}$, and the partially adjusted jackknife estimator $v_{JACK}^{PA}$. The population distribution used to generate $(y_i, x_i)$’s is close to a real data set from 1988 Current Population Survey (Valliant 1993), where $x$ is the hours worked per week and $y$ is the weekly wage. Some descriptions of the data set can be found in Chen and Shao (1999).

We consider simple random sampling with $n = 100$ or 200 and a single imputation class. The $y$-respondents are generated according to the response probability function

$$P(a = 1|x) = \frac{\exp(\gamma_1 + \gamma_2x)}{1 + \exp(\gamma_1 + \gamma_2x)}$$

with various $\gamma_1$ and $\gamma_2$. When $\gamma_2 = 0$, respondents are generated with equal probability (uniform response); when $\gamma_2 \neq 0$, response rate depends on the value of $x$ (non-uniform response). When uniform response is considered, the response rate is chosen to be between 0.5 and 0.88. Table 1 provides values of $\gamma_1$, $\gamma_2$, the ranges of $P(a = 1|x)$, and the average response rate $E[P(a = 1|x)]$.

Table 2 lists 10,000 Monte Carlo simulation estimates of the relative bias (in %) and the standard deviation for five jackknife variance estimators under 15 different models (different values of $\gamma_1$ and $\gamma_2$) described in Table 1. The values of the asymptotic variance $V$ in (4) are also listed.

The results in Table 2 can be summarized as follows.

1. The naive jackknife variance estimator $v_{JACK}$ has a serious negative bias. The relative bias of $v_{JACK}$ can be as high as 50-70% and, as expected, is related to the average response rate $E[P(a = 1|x)]$. The re-imputed (or adjusted) jackknife variance estimator $v_{JACK}^R$ has a serious positive bias, as indicated by Theorem 1. The relative biases of $v_{JACK}$ and $v_{JACK}^R$ are comparable in absolute value, but with different signs. Furthermore, the relative biases of $v_{JACK}$ and $v_{JACK}^R$ do not change much as $n$ increases from 100 to 200.

2. The relative biases are small for the two partially re-imputed jackknife variance estimators, $v_{PRI}^R$ and $v_{PRI}$, and the partially adjusted jackknife variance estimator $v_{JACK}^{PA}$.

3. The standard deviations of $v_{JACK}^R$, $v_{PRI}^R$, and $v_{JACK}^{PA}$ are comparable and the standard deviation of $v_{JACK}^{PA}$ is slightly smaller. All standard deviations decrease substantially as $n$ increases from 100 to 200, which supports the consistency result established in Theorem 2. The standard deviation of the naive jackknife variance estimator $v_{JACK}$ is much smaller than those of other jackknife variance estimators, but this does not indicate a good performance of $v_{JACK}$ since $v_{JACK}$ has a large negative relative bias.

**Appendix**

**Lemma 1.** Under the conditions in Theorem 1, (11) holds.

**Proof.** From Theorem 2 in Chen and Shao (1999), $E(\hat{\psi}_k - \mu_k)^2 = O(n^{-1})$, under the conditions of Theorem 1. Hence,

$$E \left[ \sum_{j \in R_k} \left( \frac{w_j \psi_j - \hat{\psi}_k}{m_k - 1} \right)^2 \right] = o(n^{-1}).$$

Similarly,

$$E \left[ \sum_{j \in R_k} \left( \frac{\bar{\psi}_k}{m_k - 1} \right)^2 \right] = o(n^{-1}).$$

Thus, the result follows from

$$E \left[ \sum_{j \in R_k} w_j \left( c_j(x_j) - \psi_j \right)^2 \right] = o(n^{-1}).$$

Let $b_j^{(k)} = \sum_{t \in N_x} d_{jt}$. Since

$$w_j c_j^{(k)} \leq w_j d_j^{(k)} = \sum_{t \in N_x} w_t d_{jt} \leq O(n^{-1}) b_j^{(k)},$$

the result follows from

$$E \left[ \sum_{j \in R_k} (b_j^{(k)})^2 \left( \psi_j(x_j) - \psi_k(x_j) \right)^2 \right] = o(n).$$

Let $\text{Var}_j$ be the conditional variance, given $x_j, j \in R_k$. Since $b_j^{(k)}$ is conditionally binomial, we have

$$E \left[ \sum_{j \in R_k} \text{Var}_j \left( b_j^{(k)} \left( \psi_j(x_j) - \psi_k(x_j) \right) \right) \right]$$

$$\leq E \left[ 2m_k \sum_{j \in R_k} \left( \psi_j(x_j) - \psi_k(x_j) \right)^2 [F_k(x_j) - F_k(x_{j_1})] \right]$$

$$= E \left[ 2m_k \int_{x < y} [\psi_k(y) - \psi_k(x)]^2 [F_k(y) - F_k(x)] \right]$$

$$\left[ 1 + F_k(x) - F_k(y) \right] f_k(x) f_k(y) \, dx \, dy.$$
Table 1: Parameters in Response Models

\[ P(a = 1|x) = \exp(\gamma_1 + \gamma_2 x)/[1 + \exp(\gamma_1 + \gamma_2 x)] \]

| Model | \( \gamma_1 \) | \( \gamma_2 \) | \( \min_x P(a = 1|x) \) | \( \max_x P(a = 1|x) \) | \( E[P(a = 1|x)] \) |
|-------|----------------|----------------|-------------------|-------------------|-------------------|
| 1     | 0              | -0.02          | 0.12              | 0.50              | 0.32              |
| 2     | 0              | -0.01          | 0.27              | 0.50              | 0.41              |
| 3     | 0              | 0.00           | 0.50              | 0.50              | 0.50              |
| 4     | 0              | 0.01           | 0.50              | 0.73              | 0.59              |
| 5     | 0              | 0.02           | 0.50              | 0.83              | 0.68              |
| 6     | 1              | -0.03          | 0.12              | 0.73              | 0.46              |
| 7     | 1              | -0.02          | 0.27              | 0.73              | 0.56              |
| 8     | 1              | -0.01          | 0.50              | 0.73              | 0.65              |
| 9     | 1              | 0.00           | 0.73              | 0.73              | 0.73              |
| 10    | 1              | 0.01           | 0.73              | 0.88              | 0.80              |
| 11    | 2              | -0.04          | 0.12              | 0.88              | 0.61              |
| 12    | 2              | -0.03          | 0.27              | 0.88              | 0.70              |
| 13    | 2              | -0.02          | 0.50              | 0.88              | 0.77              |
| 14    | 2              | -0.01          | 0.73              | 0.88              | 0.83              |
| 15    | 2              | 0.00           | 0.88              | 0.88              | 0.88              |

where \( r_k \) is the size of \( \mathcal{R}_k \), \( F_k \) is the conditional distribution of \( x \) given \( a = 0 \) in imputation class \( k \) and \( f_k \) is the density function of \( F_k \). Note that \( r_k[F_k(y) - F_k(x)][1 + F_k(x) - F_k(y)]^{r_k} \) has an upper bound independent of \( r_k \). Also note that \( \psi(x) \). Hence, with the moment assumption,

\[
\tau_k \int_{y \leq x} [\psi_k(y) - \psi_k(x)]^2 [F_k(y) - F_k(x)] [1 + F_k(x) - F_k(y)] dx dy < \infty.
\]

Since the integrand converges to 0 almost surely when \( \tau_k \to \infty \), by the dominate convergence theorem, the integral converges to 0 as \( \tau_k \to \infty \). Therefore,

\[
E \left[ \sum_{j \in \mathcal{R}_k} \text{Var}(b_j^{(k)}[\psi_k(x_j) - \psi_k(x_j)]) \right] = o(n)
\]

since \( m_k = O_p(n) \). Let \( E_* \) be the conditional expectation, given \( x_j, j \in \mathcal{R}_k \). The same argument can be used to establish

\[
E \left[ \sum_{j \in \mathcal{R}_k} \{E_*(b_j^{(k)}[\psi_k(x_j) - \psi_k(x_j)])\}^2 \right] = o(n).
\]

This completes the proof.

**Lemma 2.** Under the conditions of Theorem 1,

\[
E \left( \sum_{i \in \mathcal{R}_k} d_i^{(k)} w_i^2 z_i^2 \right) = E \left( \sum_{i \in \mathcal{R}_k} w_i^2 x_i^2 \right) + o(n^{-1}),
\]

where \( z_i = \psi_k(x_i) - \mu_k \), and

\[
E \left[ \sum_{i \in \mathcal{R}_k} \left( \sum_{j \in \mathcal{N}_k} \sum_{i \in \mathcal{R}_k} w_i^2 d_{ij} t_i \right) \right] = E \left( \sum_{i \in \mathcal{R}_k} d_i^{(k)} w_i^2 t_i \right)
\]

\[+ o(n^{-1}),\]

where \( t_i = z_i^2 \) or \( \text{Var}(y|x_i) \).

**Proof.** The first result follows directly from Theorems 1 and 2 in Chen and Shao (1999). For the second result, note that

\[
E \left[ \sum_{i \in \mathcal{R}_k} w_i^2 t_i + \sum_{i \in \mathcal{R}_k} \left( \sum_{j \in \mathcal{N}_k} w_j^2 d_{ij} \right) t_i \right]
\]

\[= E_*E_m \left[ \sum_{i \in \mathcal{R}_k} a_i w_i^2 E_m(t_i|a_i = 1) \right]
\]

\[+ \sum_{i \in \mathcal{R}_k} (1 - a_i) w_i^2 \sum_{i \in \mathcal{R}_k} d_{ij} t_i \]

\[= E_* \left( \sum_{i \in \mathcal{R}_k} w_i^2 p_k E_m(t_i|a_i = 1) \right)
\]

\[+ \sum_{i \in \mathcal{R}_k} w_i^2 (1 - p_k) E_m(t_i|a_i = 0) + o(n^{-1}) \]

\[= E_* \left( \sum_{i \in \mathcal{R}_k} w_i^2 E_m(t_i) \right) + o(n^{-1}),\]

where \( E_m \) is the expectation with respect to model under assumption A, \( E_* \) is the expectation with respect to sampling, \( p_k = E(a_i) \) for \( i \) in imputation.
Table 2: Relative Bias (RB) in % and Standard Deviation (SD) of Jackknife Variance Estimations

\[ n = 100 \]

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class $k$, and the second equality follows from Theorem 1 in Chen and Shao (1999). The result then follows from the first result and $\sum_{i \in S_k} w_i^2 E_m(t_i) = \sum_{i \in R_k} w_i^2 E_m(t_i) + \sum_{i \in A_k} w_i^2 E_m(t_i)$.

**Proof of Theorem 2.** Result (14) has already been established in the derivation of $v_{\text{JACK}}^{\text{pre}}$ in Section 4. To show (15), it suffices to show that $\text{Var}(\hat{y}^{\text{pre}}_{\text{JACK}}) = O(n^{-3})$. From (9) and (10), a straightforward calculation shows that

$$\text{Var}[(\hat{y}_{\text{NNI}} - \hat{y}^{(j)}_{\text{NNI}})^2] = O(n^{-4})$$

uniformly in $j$ and

$$\text{Cov}[(\hat{y}_{\text{NNI}} - \hat{y}^{(j)}_{\text{NNI}})^2, (\hat{y}_{\text{NNI}} - \hat{y}^{(l)}_{\text{NNI}})^2] = O(n^{-5})$$

uniformly in $j \neq l$. This proves the result.

### 6 References


