DEALING WITH WIDE WEIGHT VARIATION IN POLLS

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Introduction

It is well known by survey practitioners that wide weight variation is not good. If a respondent with a relatively large weight is an outlier the resulting estimate may be inappropriately skewed. However, weight variation is not always bad. Unequal probabilities result in unbiased estimates that have weight variation. Consider optimal allocation in stratified sampling when the variance of the characteristic of interest is different from one stratum to the next. The optimal allocation results in an unbiased minimum variance estimate that has weight variation. Why do we consider wide weight variation bad? The problem is that most surveys have multiple purposes and many estimates are computed other than those for which the sample design was targeted. Some of these estimates may be for subsets of the total population so that the optimal allocation for the total population does not produce optimal results. There may be tabulations from a stratified design that overlap strata producing biased estimates with too much weight variation. Statistically the problem is to choose the estimation procedure that produces the lowest mean square error (the variance of an estimate plus the square of its bias). A biased estimate can have lower mean square error than an unbiased estimate if its variance is enough lower than the variance of the unbiased estimate to compensate for the bias.

Variance considerations

If the sampling fractions of a stratified sample diverge from optimum allocation, the variance is increased. Since in most surveys there is interest in several population characteristics which may vary in their between strata variance relationships, for variance considerations we will look at the situation when the optimum allocation would be proportional (equal variance in each stratum). Proportional allocation results in unbiased weights which are the same in every stratum. If disproportionate weights are used, say, to correct for differential nonresponse, the variance of an estimate is increased. For any fixed spread from the lowest to highest weights and for any fixed smallest weight, the maximum loss in reliability of estimates is attained when all of the remaining weight is subject to the highest weight (Kish, 1965). Thus, for any fixed ratio of the largest stratum weight to the smallest stratum weight the greatest loss in efficiency occurs when there are only two strata. Thus the extreme case of two strata is considered here.

For sample balancing each stratum receives a weight that is equal to the proportion of the population that is in the stratum divided by the proportion of the sample that is in the stratum. The resulting weights are such that the weighted sample size is equal to the actual sample size. In most polls the sample is generated to be proportional by demographic groups (equal probability) but ends up disproportional due to differential nonresponse. Consider two demographic post strata with a ratio of the larger balancing weight to the smaller balancing weight of k. In addition denote the proportion of the sample that is in the stratum with the larger balancing weight as a. The ratio of the variance using the balancing weight to the variance of the unweighted sample mean is given by (Hanson, 1969):

$$R = (1+a(k^2-1))/(1+a(k-1))^2.$$

The maximum value of R occurs when $a = 1/(k+1)$. For this value of a, $R = (k+1)^2/4k$. Table 1 gives the maximum a and R values for selected k values.

Table 1

k	Max. a	Max. R
1.5	0.4	1.041
2.0	0.333	1.125
3.0	0.25	1.333
5.0	0.167	1.8
7.5	0.118	2.408
10.0	0.091	3.025
15.0	0.0625	4.267
20.0	0.048	5.513
25.0	0.038	6.76

It can be shown that if a is at its maximum value and the balancing weight for stratum 2 is k times the balancing weight for stratum 1, then the proportion of the population is the same in both strata.

Mean Square Error Analysis

The variance considerations are not of much practical use since it will almost never be the case that the variance of a item of interest is the same in both strata. Due to differential nonresponse, the unweighted estimate is biased. Consider two post strata each comprising one half of the population and a simple random sample of size 1000 selected from the entire population. The responding sample of size 1000 is post stratified into the two strata. The resulting sample in each stratum is a conditional random sample from that stratum. For this analysis the resulting sample in each stratum will be considered fixed and not treated as a random variable. Due to differential nonresponse the sampling rates in the two strata are not the same. For the unbiased estimate over the two strata the ratio of the stratum 2 weight to the stratum 1 weight is denoted as j. This estimate uses the balancing weight calculated by dividing the stratum population proportion, .5, by the stratum sample proportion. We want to pick a value k and change the balancing weights so the ratio of the weights is reduced to k subject to the constraint that the weighted sample size remains at 1000.

Denote:

n1 + n2 = n = 1000; stratum sample sizes
Wi = .5 = proportion of population in stratum i, i = 1,2
w1 = stratum 1 weight = .5/(n1/1000)
w2 = stratum 2 weight = .5/(n2/1000)
pi = proportion of stratum i population with the characteristic of interest; qi = 1-pi

For a given j, jw1 = w2 implies that n1 = (jW1n/(W2+jW1)).

The variance of the unbiased estimate, Pu, is given by:

$$Var(Pu)= (n1w1^2p1q1+n2w2^2p2q2)/n^2.$$

We pick a k < j so that w2' = kw1' where w1' and w2' are adjusted weights for stratum 1 and 2 such that n1w1' + n2w2' = n = 1000. As a result:

w1' = n/(n1+n2k).

The biased estimate, Pb, uses the weights w1' and w2'. We have:

Bias(Pb)= p1(w1'n1/n-W1)+p2(w2'n2/n-W2), and

$$Var(Pb)= (n1w1'^2p1q1+n2w2'^2p2q2)/n^2.$$

The Mean Square Error (MSE) of Pb = Var (Pb) + (Bias (Pb))^2.
We are interested in the ratio, R = MSE (Pb)/ Var(Pu). If R is less than 1 the reduction in variance in Pb resulting from reducing the weight ratio is not offset by too much bias so that Pb is better. If R is greater than 1 then the increase in bias is too much and Pu is better.

The ratio, R, of the mean square error of the biased estimate with weight ratio of 10 to the variance of the unbiased estimate with weight ratio of 15 was computed for 36 combinations of the proportion of persons in stratum 1 with the characteristic of interest and the proportion of persons in stratum 2 with the characteristic of interest between .4 and .65. For example, if the stratum 1 proportion (p1) is .45 and the stratum 2 proportion (p2) is .5, this ratio is .713. This means that the biased estimate produces an estimate with about 29 percent less mean square error than the unbiased estimate. On the other hand, if the stratum 1 proportion is .45 and the stratum 2 proportion is .65, the ratio is 1.10; the unbiased estimate is better. When the difference between p1 and p2 is high, a given level of weight reduction produces a higher mean square error ratio (less improvement by weight reduction) due to a greater concern with bias. Note that the ratio of the mean square errors is the same if P1 = P2. This is because if the proportions are equal, both estimates are unbiased and the within stratum variances are identical. The average R value over all combinations of p1 and p2 considered for a unbiased weight ratio of 15 and a biased weight ratio of k = 10 was .831.

This entire process was repeated for all possible integer k values down to 1. The unbiased estimate weight ratio remained fixed at 15. Then everything was repeated for unbiased estimate weight ratios of 25, 20, 10, 7.5, and 5. The proportion of the sample in stratum 1 was changed appropriately each time to produce the desired unbiased weight ratio. Table 2, below, gives the results in terms of the k value that produced the lowest average ratio of the mean square errors over the 36 combinations of P1 and P2. The biased estimate using this integer weight ratio produces an estimate with a lower mean square error than any higher or lower integer k value less than or equal to j. This is because there are variance improvements winning out over bias up to a point where the change in weights becomes enough to result in to much bias.

Table 2

Unbiased Weight Ratio j	Optimal Weight Ratio k	Minimum MSE Ratio R
25	14	.738
20	12	.783
15	10	.831
10	8	.893
7.5	6	.921
5	4	.969

Model Based Considerations

Consider a two strata model with i = 1,2 indicating strata and j = 1 to ni indicating sample size. Suppose yij given i are distributed independently as Bernouilli random variables with mean pi and variance piqi. Further suppose that pi ~ (po, σ^2) also independently.

Now Var($\sum$yij/ni | i) = piqi/ni (denote this Vybari). Also

$$E(p1-p2)^{\wedge}2 = Var(p1-p2) = Var(p1) + Var(p2) = 2\sigma^2$$

which implies that (p1-p2)^2/2 is an unbiased estimate of σ^2.

Stokes (1989) suggests a shrinkage estimator for the case of equal sample size in the strata. She states that there is no conceptual difference in finding shrinkage estimators when sample sizes vary among strata. Ghosh and Meeden (1986) provide suggestions for the case of different sample sizes.

Based on the ideas of Stokes (1989) let wsi denote the shrinkage weight for stratum i while as before wi denotes the unbiased weight. Then

wsi = Bi + (1 -Bi)wi with

Bi=

$$Vybari/(Vybari+(p1-p2)^2/2).$$

The amount of shrinkage is determined by the relative variability of ybari and pi. When Vybari is small compared with σ^2, sample units in the ith stratum will retain most of the weight they would for the unbiased estimate. On the other hand, if σ^2, which describes the expected diversity of strata means, is very small relative to Vybari, sample weights in the ith stratum will be shrunk to almost 1 (the unweighted mean over the strata effectively uses weights of 1 in all strata for no weight variation). The sum of the shrinkage weights over all the sample in the two strata (n1ws1 + n2ws2) will not equal n unless B1 = B2. If this is desired all shrinkage weights can be multiplied by the appropriate constant. This would not change the ratio of the weights.

The ratio of the stratum 2 shrinkage weight to the stratum 1 shrinkage weight for the same combinations of p1 and p2 examined in the Mean Square error analysis above was computed for an unbiased weight ratio of 25. For example, when p1 is .6 and p2 is .55 the ratio of the shrinkage weights is about 4.92. The average ratio over all 36 combinations was 10.1. In addition the average ratio excluding the 6 combinations for which p1 =p2 was 12.0. The later is shown since if the stratum means are the same the optimal shrinkage weight is 1 in both strata which has a large decreasing effect on the average shrinkage weight and is unlikely to happen very often in practice.

This was repeated for unbiased weight ratios of 20, 15, 10, 7.5, and 5. Results are shown in Table 3.

Table 3

Ratio of Shrinkage Weights

Unbiased Weight Ratio j	Overall Average	Average excluding p1=p2
25	10.1	12.0
20	8.9	10.5
15	7.4	8.7
10	5.7	6.6
7.5	4.6	5.3
5	3.4	3.9

The ratios in the last column of Table 3 are similar to the optimal weight ratios of Table 2.

Conclusion

The amount of weight variation that is optimal for a equal probability poll depends on the amount of differential nonresponse (unbiased weight ratio) and strata differences in characteristics of interest. Although many practitioners would like a magic number, it appears that the optimal weight ratio depends on these parameters. If stratum proportions are in the ranges studied here then the largest unbiased weight ratio for any pair of strata can be used to find an optimal weight ratio. However, if stratum differences are greater, more weight variation is probably called for.

References

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