Reporting Complete Nonresponse in a Sample Survey [1]

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1 INTRODUCTION

A great deal is written about adjusting survey estimates for both complete and item nonresponse. This paper discusses reporting complete nonresponse in sample surveys with the emphasis on reporting complete nonresponse to the survey user. Unit nonresponse is a synonym for complete nonresponse. The manager of a sample survey should know prior to enumeration who an acceptable respondent is and what constitutes a response. The survey manager has an obligation to measure complete nonresponse and to report its impact on estimates of survey parameters.

Complete nonresponse is one measure of the quality of survey data. It reflects both the difficulty of the data collection process and the competency of the organization collecting the data. Reporting complete nonresponse is a dilemma every survey organization deals with differently. How a survey organization reports survey quality is as much a test of its ethics as it is a test of its technical competence.

Reporting and measuring complete nonresponse to survey users is based on three ideas: (1) Actual survey responses are better than adjusted responses, (2) Information on the nature of complete nonresponse increases the information content of the survey, and (3) The quality of a response differs by domain-of-study. Taken together, these ideas require that complete nonresponse for important groups be reported to the user.

A list of basic references on measuring and reporting nonresponse is given in the references section. The three-volume series of the Panel of Incomplete Data (Madow, Nisselson and Olkin 1983, Madow, Olkin and Rubin 1983, Madow and Rubin 1983) provides a review of the issues of complete nonresponse and a detailed set of references.

2 COMPLETE NONRESPONSE

Each survey needs a clear definition of complete nonresponse. In this paper a complete nonresponse is a questionnaire that is either not completed or is completed by an individual not considered qualified. Alternatively, a complete response is a questionnaire completed by a qualified individual. A questionnaire is considered complete if there is enough information on the questionnaire to classify it in the surveyed population. A qualified respondent is an individual who is among a group of individuals classified as acceptable before data collection begins.

1 This paper reports the general results of research undertaken by Census Bureau staff. The views expressed are attributable only to the author and do not necessarily reflect those of the Census Bureau.

3 REPORT ON COMPLETE NONRESPONSE

For any survey, a report on complete nonresponse should be prepared at least for internal use. Survey organizations should consider publishing this information along with the survey results. In some survey organizations, there is great controversy over, and sensitivity to providing information on survey quality to survey users. This report should include:

- o Definition of Survey Universe.
- o Definition of Complete Nonresponse.
- o Description of Effect of Complete Nonresponse.
- o Description of Complete Nonresponse Adjustment.
- o Assumptions.
- o Measurement of the Impact of Complete Nonresponse.

3.1 Definition of Survey Universe

A survey report should define the target universe and each important domain-of-study. The survey universe and its domains-of-study are the populations about which survey inferences are made and conclusions drawn. The survey universe is the reference point for any conclusion. The number and characteristics of complete nonrespondents take on importance only relative to the survey population. The survey universe is the denominator for many common measures of complete nonresponse. (For examples the percent of people not responding in a survey, or the percent of the total employees in businesses not responding).

3.2 Definition of Complete Nonresponse

A report should give a clear definition of both a survey respondent and a nonrespondent. The number and characteristics of complete nonrespondents in the population or domain-of-study is the numerator in many measures of complete nonresponse. Complete nonresponse is not a black or white issue. It depends on (a) the return of the questionnaire, (b) enough information to determine whether the questionnaire is within the scope of the survey, and (c) information from an acceptable source.

A response is a questionnaire completed by a qualified respondent. A report of complete nonresponse should include an explanation of who is considered a qualified respondent and what is considered a completed questionnaire. For the purpose of this paper, a nonresponse is a survey questionnaire that is either not returned or returned without enough information to determine whether the questionnaire and the organization it represents is in the target universe.

3.2.1 Qualified Respondent.--One key aspect of complete nonresponse is the source of the information. For a questionnaire to be considered a response it must be completed by a "qualified" person. The definition of who is a "qualified" person should be included in the basic survey design. It is a decision made at the time the survey is planned.

The effect of requiring a questionnaire to be completed by a qualified respondent is that information received from "unqualified" respondents is not used. This means there is no difference between receiving information from an "unqualified" respondent and not receiving the information at all.

Conceptually there is a "continuum" of possible respondents is a survey. The survey manager and survey sponsor must decide where the dividing line is between qualified and unqualified. Individuals on one side of the dividing line are considered "qualified" and their responses are tabulated; individuals on the other side are considered "not qualified" and their responses are not tabulated.

Possible Response Continuum Survey of Household Income		
Adult with Greatest Wages Other Adult Wage Earner Adult non-wage Earner Teenager	↑	Qualified Respondent
Child Next Door Neighbor Other Neighbor Local Merchant or Official Survey Analyst's Judgement	↓	Not A Qualified Respondent

The definition of qualified will differ depending on the purpose of the survey. A person considered knowledgeable about the number of people living in a household may not be considered knowledgeable about the source of household income or how household members intend to vote in an election.

3.2.2 Completed Questionnaire.-- A complete nonresponse is defined as a survey questionnaire that is either not returned or returned without enough information to determine whether the questionnaire is in the target survey population. There are several reasons why questionnaires are not completed: (1) not received, (2) passive nonresponse, and (3) active Nonresponse.

3.2.3 Need For a Clear Definition of Complete Nonresponse.--There is a definite need for a clear, unambiguous definition of complete nonresponse. Different people have different concepts of who should be counted as a respondent. An unambiguous definition avoids confusion and misunderstanding. Survey organizations disagree, for example, on whether to consider questionnaires returned by post office as undeliverable (Postmaster Returns) or questionnaire based on information received from secondary sources as complete nonresponses. A clear definition minimizes confusion.

Postmaster Returns.--Some survey organizations consider postmaster returns (PMR) as complete responses and exclude them from any calculation of complete nonresponse. The argument for this is that the questionnaire was mailed and returned. In extreme cases, counting a PMR as a complete response distorts the measurement of complete nonresponse. The following hypothetical example for a sample of 1000 cases illustrates this.

Hypothetical Example
Postmaster Return Classified as a Respondent

Questionnaires	Number
Return Completed	100
Postmaster Return (PMR)	800
Complete Nonresponse(remainder)	100
Total Mailed	1,000
Percent Complete Nonresponse	
Counting PMRs as Nonresponse	90 Percent
Not Counting PMRs as Nonresponse	10 Percent

In practice the proportion of questionnaires returned because they cannot be delivered by post office is usually small. The actual number and effect of PMRs is a function of the survey frame and data collection procedure. Excluding PMRs from the definition of complete nonresponse changes the reported percent of complete nonresponse.

Secondary Sources.--Some survey organizations make no distinction concerning the source of information. Information available directly from the desired respondent is often collected from related or secondary sources. A secondary source is someone who has knowledge of the survey unit, but does not have any direct connection. In a household survey this may be a neighbor; in a business survey, it may be the owner of another business, trade organization officer, or a government official. Once data is collected, a distinction is rarely made about where the data came from. Secondary source information may make up a small proportion of the total population but a large proportion of a domain-of-study. Often information obtained from secondary sources are for large sample units. A clear definition of what constitutes a complete response or nonresponse avoids confusion and misunderstanding in decision making.

3.3 Description of the Effect of Complete Nonresponse

Any survey report should include a description of the potential effect of complete nonresponse on survey estimates and the conclusions based on survey results. This description should indicate that when no adjustment is made for complete nonresponse that: (1) estimates of parameters may be biased, (2) measures of precision may be biased, and (3) tests of hypotheses may be biased. The direction and magnitude of the bias of both estimates and measures of precision will depend on the variable, the parameter, and the domain-of-study. Complete nonresponse may affect the estimates of sampling error and confidence intervals. In general, tests of population parameters will be biased. The difference between actual and expected type I and type II errors will depend on the amount of bias of both estimates and the precision of those estimates.

3.3.1 <u>Population Parameters and Their Estimates Under Simple Random Sampling with Replacement</u>.--A population universe contains N population elements X_i i=1, 2, ..., N. A sample of n sample units x_i i i=1,2,...n is selected from the N population units. The sample is assumed to be an independent and identically distributed sample selected from a normal distribution. The population and sample parameters are given below. The sample of n sample units contains n_n nonrespondents and n_r respondents. A test of the null hypothesis that the population mean, μ, is equal to μ_o against the two-sided alternative hypothesis is done using a student t-test of the form:

$$t_{n-1} = [\hat{\mu} - \mu_o] / [\hat{\sigma} / \sqrt{n}]$$

The effect of complete nonresponse and the adjustment for complete nonresponse is discussed in the context of these parameters and tests.

3.3.2 <u>Effect of Complete Nonresponse on Estimates of Parameters From a Simple Random Sample With Replacement</u>.--In a sample survey, an attempt is made to collect information for each survey unit selected into the sample. Often, however, it is not possible to collect information from each sample unit. Ultimately each sample unit is classified either as a respondent or a complete nonrespondent.

In some surveys, only information from respondents is used to make estimates of the universe parameters. Using information from only respondents may bias the results. The bias associated with using information only from respondents and treating it like it was a smaller sample is given in Table 1.

<u>Effect of Complete Nonresponse on Total, Mean, and Variance.</u>--Without adjustment for complete nonresponse, estimates of parameters are usually biased. The amount of the bias depends on the relative characteristics of responding and nonresponding sample units. This is seen by examining the bias terms in the table above. The more complex the population parameter, the more factors that influence the bias term. The bias of the estimate of the population total and variance illustrates this. When nonresponding sample units are smaller than the responding sample units, estimates of the mean and total have a positive bias. When nonresponding sample units are larger than the responding sample units, estimates on the population mean and total have a negative bias. The bias of the estimate of the population variance depends on both the size and the variation of respondents and nonrespondents.

<u>Effect of Complete Nonresponse on Hypothesis Testing.</u>--Complete nonresponse affects tests of hypotheses. One example is the student t test of the hypothesis that a population mean is equal to a specific value. Under the assumption, that sample comes from a normal distribution, the null hypothesis, alternative hypothesis, and test of significance using only respondent data is given below.

$$H_o : \mu = \mu o$$

$$H_a : \mu \neq \mu o$$

Reject the null hypothesis if

$$t > t_{,n-1} \text{ or } t < - t_{,n-1}$$

where

$$t = (\hat{\mu}_r - \mu_o) / (\hat{\sigma}_r / \sqrt{n_r})$$

$$= \{(\hat{\mu}_r - \mu_r) + \Theta\} / (\hat{\sigma}_r / \sqrt{n_r})$$

When the null hypothesis is false, the test statistic t has a noncentral t distribution with noncentrality parameter, Θ.

where

$$\Theta = [(\mu_r - \mu) + (\mu - \mu_o)]$$

$$= [(N_n/N)(\mu_r - \mu_n) + (\mu - \mu_o)]$$

The effect of complete nonresponse on the test of the null hypothesis is seen in the additional

Table 1. Bias Due to Complete Nonresponse of Respondent Estimates of Universe Parameters

Parameter		Estimator	Bias *
Mean	$\mu = \sum_{i}^{N} X_i / N$	$\hat{\mu}_r = \sum_{i}^{n_r} {}_r x_i / n_r$	$[N_n/N][(\mu_r - \mu_n)]$
Total	$X = \sum_{i}^{N} X_i$	$\hat{X} = N \hat{\mu}_r$	$N_n(\mu_r - \mu_n)$
Variance	$\sigma^2 = \sum_{i}^{N} (X_i - \mu)^2 / N$	$\hat{\sigma}^2 = [\sum_{i}^{n_r} ({}_r x_i - \bar{x})^2] / n_r$	$[N_n/N][\sigma_r^2 - \sigma_n^2] - [N_r/N][N_n/N][\mu_r - \mu_n]^2$

* The bias of the mean and total are given in Cochran (1977). The bias of the variance is given in Appendix A.

factor $(N_n/N)(\mu_r-\mu_n)$in the noncentrality parameter. When there is complete nonresponse, the t statistic has a noncentral t distribution. The test of the null hypothesis is clearly affected by complete nonresponse.

Effect of Complete Nonresponse on Confidence Limits. -- A 1- α confidence interval about the mean is given below.

$$\{[\hat{\mu}_r-t_{\alpha/2}(\hat{\sigma}_r/\sqrt{n_r})]<\mu<[\hat{\mu}_r+t_{\alpha/2}(\hat{\sigma}_r/\sqrt{n_r})]\}= 1-\alpha$$

When the sample is selected from a normal distribution and statistics use only information from respondents, the center of confidence interval, μ_r, depends on the bias of μ_r as an estimator of μ. The length of the confidence interval is affected by (a) the reduction in the sample size due to nonresponse, and (b) the bias in the estimate of the sample variance.

Complete nonresponse changes the effective sample size and estimate of the sample variance. The reduction in sample size from n to r tends to increase the length of the confidence interval. When the sample mean and variance of respondents and nonrespondents are different, the length of the confidence interval tends to decrease.

3.4 Description of Complete Nonresponse Adjustment

Any report of survey complete nonresponse should include a description of the complete nonresponse adjustment procedure. It should include a description of the methodology used to obtain a response and how the data was adjustment for complete nonresponse. An statement should be made concerning the relationship between the magnitude of the adjustment for complete nonresponse and the magnitude of the actual effect of complete nonresponse. An explanation should be given that the procedure for adjusting for complete nonresponse hopes to increase the accuracy of the survey by correcting for differential complete nonresponse. It should be acknowledged that the adjustment procedure can introduce additional errors into the survey and decrease the accuracy of estimates.

Adjustment for complete nonresponse is usually done by either changing the survey weights of responding sample units or substituting complete respondent records to represent nonresponding sample units. Each method of adjustment has different costs and statistical characteristics. The object of each adjustment procedure is to make the statistical characteristics of the adjusted sample the same as the sample would have been had there been no complete nonresponse.

The adjustment method selected is influenced by the information available, the data collection system, and the data processing system. Failure to adjust for complete nonresponse, in general, introduces a bias into estimates of population parameters. "Proper" adjustment for complete nonresponse may eliminate or reduce that bias. "Improper" adjustment may add to the already existing bias.

If the adjustment is good, both the expected value of the mean and the variance of the adjusted population are close to the mean and variance of the nonresponding population. It is not the magnitude of the adjustment that is important; it is how close the expected value of an adjusted value is to the expected value of a complete nonrespondent. It is possible to have: (1) zero bias and a "high" rate of complete nonresponse, (2) zero bias and a "high" rate of adjustment, (3) high bias and a "low" rate of complete nonresponse, and (4) high bias and a "low" rate of adjustment.

These relationships make the decision on how to measure the effect of complete nonresponse difficult. No one easily calculated statistic accurately reflects the effect complete nonresponse has had on the estimates. Any report on complete nonresponse should identify the differences between the effect of complete nonresponse and the effect of the adjustment for complete nonresponse.

3.5 Assumptions

Any report describing the adjustment for complete nonresponse should state the assumptions used in the adjustment methodology. Assumptions about the relationship between respondents and nonrespondents are often made when adjusting for complete nonresponse. Frequently these assumptions must be true before the estimator has desirable statistical properties such as unbiasedness. A statement of all assumptions used in the adjustment for complete nonresponse should be included in the description of the survey methods. Stating each assumption and the effect of not meeting allows the user to evaluate the appropriateness of the assumption, and to decide whether multiple assumptions are consistent, and the likelihood that they will be met.

In some surveys the primary objective of complete nonresponse adjustment is for the sample to be unbiased. For this to occur, some surveys require the assumption that nonrespondents and respondents have the same expected value. With more complex statistics or more sophisticated adjustment procedures, two or more assumptions must be met for an objective such as an unbiased estimate to be achieved.

3.5.1 Single Assumption--If a simple random sample of sample size n is selected from a universe of N, then an estimator of the population total is the respondent mean times the universe size and is given by

$$\hat{X} = N\,\hat{\mu}_r .$$

For this to be an unbiased estimator of the population, one assumption is necessary: the expected value of the sample mean must be the same for responding and nonresponding sample units. When this assumption is not true, the estimator of the population total may not be unbiased.

3.5.2 Multiple Assumptions.--In some surveys, adjustment for complete nonresponse requires multiple assumptions. In many surveys, sample units are classified as either in-scope or out-of-scope to the survey. Only sample units classified as in the survey are tabulated. One example is a survey of small businesses using an out-of-date frame.

If a survey obtained one hundred percent response, only the sample units in-scope (still in business) would be tabulated. Sample cases that qualify for the survey are said to be "in-scope" to the survey. If a simple random sample of sample size n is selected from a universe of N, the estimator of the population total is given by

$$\hat{X} = N \hat{p}_r \hat{\mu}_r$$

The universe total, N, is known from the sample design. The statistic $\hat{p}_r$ estimates the proportion of respondent sample units that are in the survey population. The statistic $\hat{\mu}_r$ estimates the mean value for respondents. Information on the characteristics of responding sample units is available from the survey. No information is available on the characteristics of nonresponding sample units.

In this example, three assumptions are necessary for this estimator of the population total to be unbiased: (1) the expected value of the sample mean must be the same for responding and nonresponding sample units, (2) the expected proportion of in-scope sample units must be the same for responding and nonresponding units, and (3) the proportion of in-scope sample units must be statistically independent of the mean of in-scope sample units. Statistical independence is necessary so that the expected value of the product equals the product of the expected values. If one of these assumptions is not true, the expected result is not true. In complicated sample designs, the number of assumptions and the complexity of those assumptions required for an estimator to be unbiased increase.

These assumptions should be stated in the report that accompanies the survey estimates. Users should know the assumptions necessary for the estimates to be unbiased.

3.6 Measuring the Impact of Complete Nonresponse on Survey Estimates

Any report of survey complete nonresponse should include a measure of the effect of complete nonresponse on estimates of survey parameters. It is not always easy or meaningful to measure the real effect of complete nonresponse.

3.6.1 Impact of Complete Nonresponse on Survey Estimates--The real effect of complete nonresponse on survey estimates is in how well any adjustment for complete nonresponse corrects for the effect of complete nonresponse on survey estimates. Estimates should be made of the actual impact of complete nonresponse on estimates of the parameters of the survey universe and major domains-of-study. The "real" impact of complete nonresponse in a survey is not the amount of complete nonresponse nor the amount of the adjustment for complete nonresponse; rather, it is the difference between the expected value of the adjustment for complete nonresponse and the expected value of the effect of complete nonresponse. The "real" impact of complete nonresponse is the failure to correct for complete nonresponse.

An obvious question is "How do we measure the impact of Nonresponse?" There are three reasonable measures of the effect of complete nonresponse: (a) the amount of complete nonresponse, (b) the amount of adjustment for complete nonresponse, and (c) the difference between the adjustment for complete nonresponse and the amount of nonresponse. Each of these measures has its own special problems.

3.6.2 Problems Estimating the Impact of Complete Nonresponse--A basic assumption is that information provided by a respondent is "better" than information supplied from other sources. If we had perfect information, the bias of the estimator of a population total and mean due to complete nonresponse is the ideal measure of the effect of complete nonresponse. However, we do not have perfect information and the impact must be estimated in some fashion. It is not clear what measure should be used to report the effect of complete nonresponse on survey estimates. Under usual survey sample situations, there does not appear to be any good measure of complete nonresponse.

Effect of Complete Nonresponse.--The weighted number of complete nonrespondents and their total for characteristics is a logical measure of complete nonresponse. The proportion of nonresponding sample units and the proportion of their total for a characteristic are two measures reflecting the amount of the population associated with complete nonresponse. The percent of the frame classified as complete nonresponse can usually be determined from the survey when each sample unit is classified as either a response or complete nonresponse. When all elements in the frame are not in the survey universe, it is more difficult to estimate the percent of the universe associated with complete nonresponse. The percent of the universe total for a particular characteristic is also not easily measured. Sample units classified as complete nonresponses by definition provide no information about themselves.

Sometimes a subsample is selected and more intensive and costly data collection procedures are used to determine information on complete nonrespondents. This information may not be obtained from "hard core" nonrespondents. These "hard core" complete nonrespondents may be substantially different from the other complete nonrespondents. In some cases auxiliary information on the frame can be used to estimate the effect of complete nonresponse on a parameter. In surveys based on a census or administrative list, all sample units may have one or more auxiliary characteristics. For example the Standard Statistical Establishment List of the Bureau of the Census has employment and payroll for each establishment. Auxiliary information may act as a proxy for the actual information on both the responding and nonresponding sample units. Often the auxiliary information may be the same as the desired information but for an earlier year. If the proxy information from the frame is similar to the actual information, reasonable estimates of the amount of complete nonresponse may be made. In many frames auxiliary information is not available. In these cases, estimates of the percent of a total associated with complete nonresponse may not be possible.

Adjustment for Complete Nonresponse--The amount of the adjustment for complete nonresponse is the proportion of the survey population estimate associated with sample units adjusted for complete nonresponse. Depending on the method of adjustment, this may be easy or impossible to estimate. When complete nonrespondent records are replaced with the records of respondents, this may be straight forward. When adjustment is done by weight adjustment on responding records, the amount of the adjustment must be done by comparing the adjusted and unadjusted weights. Unless the comparison is planned in advance, both these weights may not be available. This could prevent detailed calculations and require time-consuming and costly additional work related to file matching and computation.

Measuring Real Effect of Complete Nonresponse--A reasonable question to ask is "What measure do we use to report to users and project managers?" The answer will depend on what we want to know. If our interest is in data collection, we should measure the weighted and unweighted sample units classified as complete nonrespondents. This information will tell us what sample units are not responding. If the purpose is to evaluate the effect of complete nonresponse of survey estimates of population parameters, then we should measure difference between the adjustment for complete nonresponse and the actual characteristics of complete nonrespondents. This difference will reflect how close the adjustment procedure comes to reflecting the characteristics of complete nonrespondents. Clearly different measures supply different information and have different costs to produce. Cost differences are due to different survey designs, frames, data processing, and adjustment for complete nonresponse.

4 CONCLUSION

Complete nonresponse can be a major problem in a statistical survey. It can have a clear impact on survey estimates. Survey users, in order to make decisions on the quality of survey data, need to know the quality of the data. Complete nonresponse and its effect on survey estimates is one measure of the survey quality.

There appears to be no easy or straight forward measure of the "real", after adjustment, effect of complete nonresponse. The amount of complete nonresponse is often difficult to estimate and does not consider the effect of any adjustment that has been made. The adjustment for complete nonresponse may be easy to measure but requires the assumption that all complete nonresponse was adjusted for. The "best" measure of the final effect of complete nonresponse is the difference between the amount of nonresponse and the amount of adjustment for key variables and important domains-of-study.

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Appendix A

The bias of $\hat{\sigma}^2_r$ as an estimator of the variance, σ^2.

$$\hat{\sigma}^2_r = \sum_i^{n_r} ({}_r x_i - \mu_r)^2/(n_r - 1)$$

$$E(\hat{\sigma}^2_r) = \sigma^2_r$$

$$\sigma^2 = [(N_r/N)\sigma_r^2 + (N_n/N)\sigma_n^2] + \{(N_r/N)(N_n/N)(\mu_r - \mu_n)^2\}$$

$$\begin{aligned}
Bias(\hat{\sigma}^2_r) &= E(\hat{\sigma}^2_r - \sigma^2) \\
&= \sigma_r^2 - \sigma^2 \\
&= \sigma_r^2 - \{[(N_r/N)\sigma_r^2 + (N_n/N)\sigma_n^2] + [(N_r/N)(N_n/N)(\mu_r - \mu_n)^2]\} \\
&= \{[(N_r/N)\sigma_r^2 + (N_n/N)\sigma_r^2]\} - \{[(N_r/N)\sigma_r^2 + (N_n/N)\sigma_n^2] + [(N_r/N)(N_n/N)(\mu_r - \mu_n)^2]\} \\
&= [(N_n/N)(\sigma_r^2 - \sigma_n^2)] - [(N_r/N)(N_n/N)(\mu_r - \mu_n)^2] \\
&= (N_n/N)\,[(\sigma_r^2 - \sigma_n^2) - (N_r/N)(\mu_r - \mu_n)^2]
\end{aligned}$$