ESTIMATION OF THE MEASUREMENT ERROR MODEL WITH UNEQUAL ERROR VARIANCES

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KEY WORDS: Errors-in-variables, generalized least squares

1. Introduction

The statistical consideration of models containing measurement errors began as early as 1877. See Fuller (1987, p. 30). Most of the past work has been done on the univariate linear model with constant error variances. More recently, work has been done on multivariate, non-linear, and non-constant error variance models. We will consider a methods of moments estimator for the parameters of a univariate linear model with heteroskedastic error variances.

The general univariate linear measurement error model is

$$y_t = \mathbf{x}_t\boldsymbol{\beta} + q_t\,,\ t=1,2,\ldots,n, \tag{1.1}$$

where y_t is the dependent observation at time t, $\mathbf{x}_t$ is a 1 x k vector of explanatory variables, $\boldsymbol{\beta}$ is a k x 1 vector of unknown coefficients, and q_t is the equation error. The usual goal is to estimate $\boldsymbol{\beta}$. We assume that we are unable to observe $\mathbf{z}_t = (y_t, \mathbf{x}_t)$ directly. Instead, we observe $\mathbf{Z}_t = (Y_t, \mathbf{X}_t)$, such that

$$\begin{aligned} Y_t &= y_t + w_t, \\ \mathbf{X}_t &= \mathbf{x}_t + \mathbf{u}_t,\ t=1,2,..,n, \end{aligned} \tag{1.2}$$

where $\mathbf{a}_t = (w_t, \mathbf{u}_t)$ are random measurement errors. We further assume that

$$\begin{bmatrix} \mathbf{x}_t' \\ q_t \\ w_t \\ \mathbf{u}_t' \end{bmatrix} \sim \mathrm{NI}\left[\begin{bmatrix} \boldsymbol{\mu}_{xt}' \\ 0 \\ 0 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{xx} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_{qq} & 0 & \mathbf{0} \\ \mathbf{0} & 0 & \sigma_{wwtt} & \boldsymbol{\Sigma}_{wutt} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_{uwtt} & \boldsymbol{\Sigma}_{uutt} \end{bmatrix}\right] \tag{1.3}$$

where

$$\boldsymbol{\Sigma}_{aatt} = \begin{bmatrix} \sigma_{wwtt} & \boldsymbol{\Sigma}_{wutt} \\ \boldsymbol{\Sigma}_{uwtt} & \boldsymbol{\Sigma}_{uutt} \end{bmatrix},\ t = 1,\ldots,n,$$

are known for each t. This is the heteroskedastic measurement error model with normal distribution assumptions.

If the $\boldsymbol{\Sigma}_{aatt}$ are known, a natural estimator is

$$\tilde{\boldsymbol{\beta}} = \hat{\mathbf{M}}_{xx}^{-1}\hat{\mathbf{M}}_{xy} \tag{1.4}$$

where

$$\hat{\mathbf{M}}_{xx} = n^{-1}\sum_{t=1}^{n}(\mathbf{X}_t'\mathbf{X}_t - \boldsymbol{\Sigma}_{uutt})$$

and

$$\hat{\mathbf{M}}_{xy} = n^{-1}\sum_{t=1}^{n}(\mathbf{X}_t'Y_t - \boldsymbol{\Sigma}_{uwtt}).$$

Fuller (1987, Section 3.1) showed that

$$\hat{\mathbf{V}}_{\beta\beta}^{-1/2}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{L} \mathrm{N}(\mathbf{0}, \mathbf{I}),$$

where

$$\hat{\mathbf{V}}_{\beta\beta} = n^{-1}\hat{\mathbf{M}}_{xx}^{-1}\hat{\mathbf{G}}\hat{\mathbf{M}}_{xx}^{-1},$$

$$\hat{\mathbf{G}} = n^{-1}\sum_{t=1}^{n}(\mathbf{X}_t'\mathbf{X}_t\tilde{\sigma}_{vvtt} + \tilde{\boldsymbol{\Sigma}}_{uvtt}\tilde{\boldsymbol{\Sigma}}_{vutt}),$$

$$\tilde{\sigma}_{vvtt} = \tilde{\sigma}_{qq} + \sigma_{wwtt} - 2\tilde{\boldsymbol{\beta}}'\boldsymbol{\Sigma}_{uwtt} + \tilde{\boldsymbol{\beta}}'\boldsymbol{\Sigma}_{uutt}\tilde{\boldsymbol{\beta}}, \tag{1.5}$$

$$\tilde{\boldsymbol{\Sigma}}_{uvtt} = \boldsymbol{\Sigma}_{uwtt} - \boldsymbol{\Sigma}_{uutt}\tilde{\boldsymbol{\beta}},$$

and

$$\tilde{\sigma}_{qq} = \sum_{t=1}^{n}[(n\text{-}k)^{-1}(Y_t - \mathbf{X}_t\tilde{\boldsymbol{\beta}})^2 - n^{-1}(1, -\tilde{\boldsymbol{\beta}}')\boldsymbol{\Sigma}_{aatt}(1, -\tilde{\boldsymbol{\beta}}')'].$$

The estimator $\tilde{\boldsymbol{\beta}}$ is a consistent estimator and relatively easy to compute. However, it is sometimes possible to construct an asymptotically superior estimator using $\tilde{\boldsymbol{\beta}}$ as a preliminary estimator. Hasabelnaby (1987) investigated the weighted estimator,

$$\hat{\boldsymbol{\beta}} = [\sum_{t=1}^{n}\tilde{\sigma}_{vvtt}^{-1}(\mathbf{X}_t'\mathbf{X}_t - \boldsymbol{\Sigma}_{uutt})]^{-1} \times \sum_{t=1}^{n}\tilde{\sigma}_{vvtt}^{-1}(\mathbf{X}_t'Y_t - \boldsymbol{\Sigma}_{uwtt}) \tag{1.6}$$

where $\tilde{\sigma}_{vvtt}$, an estimator of the variance of $v_t = q_t + w_t - \mathbf{u}_t\boldsymbol{\beta}$, is defined in (1.5). Hasabelnaby showed that,

$$\hat{\mathbf{V}}_{\beta\beta}^{-1/2}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) \xrightarrow{L} N(\mathbf{0}, \mathbf{I}),$$

where

$$\hat{\mathbf{V}}_{\beta\beta} = n^{-2}\hat{\mathbf{M}}_{x\pi x}^{-1}[\sum_{t=1}^{n} \tilde{\sigma}_{vvtt}^{-1}(\mathbf{X}_t'\mathbf{X}_t + \tilde{\sigma}_{vvtt}^{-1}\tilde{\boldsymbol{\Sigma}}_{uvtt}\tilde{\boldsymbol{\Sigma}}_{vutt})]\hat{\mathbf{M}}_{x\pi x}^{-1},$$

and

$$\hat{\mathbf{M}}_{x\pi x} = \sum_{t=1}^{n} \tilde{\sigma}_{vvtt}^{-1}(\mathbf{X}_t'\mathbf{X}_t - \boldsymbol{\Sigma}_{uutt}).$$

The use of $\tilde{\sigma}_{vvtt}^{-1}$ as a weight minimizes the part of the covariance matrix of the limiting distribution associated with $\mathbf{x}_t'v_t$. There is also a contribution of the variance of $\mathbf{u}_t'v_t$ to the covariance matrix. The set of weights minimizing the variance of the limiting distribution depends on the unknown $\mathbf{x}_t$. Since a consistent estimator of the $\mathbf{x}_t$ does not exist, a best set of weights can not be constructed without additional assumptions.

Under certain assumptions, it is possible to construct an estimator of $\boldsymbol{\beta}$ that is generally better than (1.6) and better than (1.4). The remainder of this paper will discuss a method of moments estimator based on a two-stage generalized least squares estimator of $\boldsymbol{\Sigma}_{zz}$ developed by Fuller (1990).

2. Estimation of the Covariance Matrix

Consider the model

$$\mathbf{Z}_t = \mathbf{z}_t + \mathbf{a}_t, \; t = 1,\dots,n \tag{2.1}$$

where $\mathbf{Z}_t$ is a p-dimensional row vector of observed values, $\mathbf{z}_t$ is a p-dimensional row vector of true values, $\mathbf{a}_t$ is a p-dimensional row vector of measurement errors, and $n > p$.

Assume that

$$\begin{bmatrix} \mathbf{z}_t' \\ \mathbf{a}_t' \end{bmatrix} \sim NI\left(\begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{zz} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{aatt} \end{bmatrix}\right), \; t = 1,\dots,n \tag{2.2}$$

where $\boldsymbol{\Sigma}_{aatt}$ is known for all t. Let

$$\boldsymbol{\Sigma}_{ZZtt} = \boldsymbol{\Sigma}_{zz} + \boldsymbol{\Sigma}_{aatt}.$$

Assume that $\boldsymbol{\Sigma}_{ZZtt}$ is nonsingular for all t. Define

$$\mathbf{m}_{ZZ} = (n-1)^{-1}\sum_{t=1}^{n}(\mathbf{Z}_t - \bar{\mathbf{Z}})'(\mathbf{Z}_t - \bar{\mathbf{Z}}),$$

$$\boldsymbol{\Sigma}_{aa..} = n^{-1}\sum_{t=1}^{n}\boldsymbol{\Sigma}_{aatt},$$

and

$$\bar{\mathbf{Z}} = n^{-1}\sum_{t=1}^{n}\mathbf{Z}_t. \tag{2.3}$$

Fuller (1989) considered the following preliminary estimator of $\boldsymbol{\Sigma}_{zz}$,

$$\hat{\boldsymbol{\Sigma}}_{zz} = \mathbf{m}_{ZZ} - \boldsymbol{\Sigma}_{aa..}. \tag{2.4}$$

Fuller also modified (2.4) to ensure that $\hat{\boldsymbol{\Sigma}}_{zz}$ is positive semidefinite. For simplicity, we will initially assume that estimator (2.4) is positive semidefinite.

Under model (2.1) - (2.2) it can be shown that

$$E(\hat{\boldsymbol{\Sigma}}_{zz}) = \boldsymbol{\Sigma}_{zz}.$$

Define

$$\hat{\boldsymbol{\Sigma}}_{ZZtt} = \hat{\boldsymbol{\Sigma}}_{zz} + \boldsymbol{\Sigma}_{aatt}. \tag{2.5}$$

This estimator is an unbiased preliminary estimator of $\boldsymbol{\Sigma}_{ZZtt}$.

Let $\mathbf{M}$ be any p x p matrix. Let vec $\mathbf{M}$ denote the p^2 column vector obtained by listing the columns of $\mathbf{M}$ one beneath the other in a single column. Let vech $\mathbf{M}$ denote the $p(p+1)/2$ column vector obtained by listing the elements in each column that are on or below the diagonal. In addition, let $\boldsymbol{\Phi}_p$ be the $p^2 \times p(p+1)/2$ matrix such that

$$\text{vec}\,\mathbf{M} = \boldsymbol{\Phi}_p \text{vech}\,\mathbf{M}.$$

See, for example, Fuller (1987, Appendix 4A) for discussion of vec, vech, and the matrix $\boldsymbol{\Phi}_p$.

Fuller considered the second round estimators given by

$$\hat{\boldsymbol{\mu}}' = [\sum_{t=1}^{n}\hat{\mathbf{W}}_{ZZtt}^{-1}]^{-1}\sum_{t=1}^{n}\hat{\mathbf{W}}_{ZZtt}^{-1}\mathbf{Z}_t', \tag{2.6}$$

$$\text{vech}\,\hat{\mathbf{m}}_{..} = n(n-1)^{-1}(\sum_{t=1}^{n}\hat{\mathbf{V}}_{mmtt}^{-1})^{-1} \times \sum_{t=1}^{n}\hat{\mathbf{V}}_{mmtt}^{-1}\text{vech}[(\mathbf{Z}_t - \hat{\boldsymbol{\mu}})'(\mathbf{Z}_t - \hat{\boldsymbol{\mu}})], \tag{2.7}$$

$$\text{vech}\,\boldsymbol{\Sigma}_{a\pi a..} = \Big(\sum_{t=1}^{n}\hat{\mathbf{V}}^{-1}_{mmtt}\Big)^{-1}\sum_{t=1}^{n}\hat{\mathbf{V}}^{-1}_{mmtt}\text{vech}\,\boldsymbol{\Sigma}_{aatt}, \tag{2.8}$$

$$\hat{\mathbf{V}}^{-1}_{mmtt} = 1/2\boldsymbol{\Phi}_p'(\hat{\mathbf{W}}^{-1}_{ZZtt}\otimes\hat{\mathbf{W}}^{-1}_{ZZtt})\boldsymbol{\Phi}_p, \tag{2.9}$$

$$\hat{\mathbf{W}}_{ZZtt} = n^{-1}(n-1)\hat{\boldsymbol{\Sigma}}_{ZZtt} + n^{-1}\mathbf{m}_{ZZ}. \tag{2.10}$$

The $\hat{\mathbf{W}}_{ZZtt}$ are preliminary estimators of $\boldsymbol{\Sigma}_{ZZtt}$. The inclusion of the $n^{-1}\mathbf{m}_{ZZ}$ term in $\hat{\mathbf{W}}_{ZZtt}$ produces a more stable estimator for $\boldsymbol{\Sigma}^{-1}_{ZZtt}$. Hence, $\hat{\boldsymbol{\mu}}$ is an estimated generalized least squares estimator of $\boldsymbol{\mu}$. Also, under the normality assumption (2.2),

$$\mathbf{V}^{-1}_{mmtt} \equiv 1/2\boldsymbol{\Phi}_p'(\boldsymbol{\Sigma}^{-1}_{ZZtt}\otimes\boldsymbol{\Sigma}^{-1}_{ZZtt})\boldsymbol{\Phi}_p$$

is the inverse of the covariance matrix of

$$\text{vech}[(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})],$$

and (2.9) is an estimator of $\mathbf{V}^{-1}_{mmtt}$. See Fuller (1987, p. 386). Thus

$$\tilde{\boldsymbol{\Sigma}}_{zz} = \hat{\mathbf{m}}_{..} - \boldsymbol{\Sigma}_{a\pi a..} \tag{2.11}$$

is the estimated generalized least squares estimator of $\boldsymbol{\Sigma}_{zz}$. For simplicity we again initially assume that (2.11) is positive semidefinite.

The estimator vech $\tilde{\boldsymbol{\Sigma}}_{zz}$ is asymptotically normal.

Theorem 1. Assume

(1) $\mathbf{Z}_t = \mathbf{z}_t + \mathbf{a}_t$, $t = 1,\ldots,n$,

(2) $\begin{bmatrix}\mathbf{z}_t'\\ \mathbf{a}_t'\end{bmatrix} \sim \text{Ind}\left(\begin{bmatrix}\boldsymbol{\mu}\\ \mathbf{0}\end{bmatrix},\begin{bmatrix}\boldsymbol{\Sigma}_{zz} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma}_{aatt}\end{bmatrix}\right)$, $t = 1,\ldots,n$,

(3) $\boldsymbol{\Sigma}_{aatt}$ is known for all t,

(4) the true value, $\boldsymbol{\theta}_0$, is in the interior of $\boldsymbol{\Theta} = \{\boldsymbol{\theta}{:}\boldsymbol{\Sigma}_{zz}(\boldsymbol{\theta})$ is positive semidefinite$\}$,

(5) there exists an convex, open set $\boldsymbol{\Omega} \subset \boldsymbol{\Theta}$ with $\boldsymbol{\theta}_0$ in the interior such that $\det[\boldsymbol{\Sigma}_{ZZtt}(\boldsymbol{\theta})] \ge c > 0\ \forall$ t and $\forall\ \boldsymbol{\theta}\in\boldsymbol{\Omega}$,

(6) $[n^{-1}\sum_{t=1}^{n}\boldsymbol{\Sigma}^{-1}_{ZZtt}(\boldsymbol{\theta}_0)]^{-1}$ is uniformly bounded,

(7) $\lim_{n\to\infty} n^{-1}\sum_{t=1}^{n}\mathbf{V}^{-1}_{mmtt} = \bar{\mathbf{V}}_I$, where $\bar{\mathbf{V}}_I$ is positive definite,

(8) $\lim_{n\to\infty} n^{-1}\sum_{t=1}^{n}\mathbf{V}^{-1}_{mmtt}\mathbf{G}_{mmtt}\mathbf{V}^{-1}_{mmtt} = \bar{\mathbf{G}}_{VV}$, where $\bar{\mathbf{G}}_{VV}$ is positive definite and $\mathbf{G}_{mmtt} = \text{Var}\{\text{vech}[(\mathbf{Z}_t-\boldsymbol{\mu})'(\mathbf{Z}_t-\boldsymbol{\mu})]\}$,

(9) $\mathbf{Z}_t$ have uniformly bounded $4+\delta$ moments.

Then

$$n^{1/2}\text{vech}(\tilde{\boldsymbol{\Sigma}}_{zz}-\boldsymbol{\Sigma}_{zz}) \xrightarrow{L} N(\mathbf{0},\boldsymbol{\Gamma})$$

where

$$\boldsymbol{\Gamma} = \bar{\mathbf{V}}_I^{-1}\bar{\mathbf{G}}_{VV}\bar{\mathbf{V}}_I^{-1}.$$

Proof. Omitted. □

Corollary 1.1 Let the assumptions of Theorem 1 hold and, in addition, assume the $\mathbf{Z}_t'$ to be normally distributed. Then

$$\boldsymbol{\Gamma} = \bar{\mathbf{V}}_I^{-1}.$$

Proof. Omitted. □

3. Estimation of Regression Parameters

Consider the following model.

$$y_t = \beta_0 + \mathbf{x}_t\boldsymbol{\beta}_1 + q_t,$$
$$\mathbf{Z}_t = \mathbf{z}_t + \mathbf{a}_t,\ t = 1,..,n, \tag{3.1}$$

where $\mathbf{Z}_t' = (Y_t, \mathbf{X}_t')'$ and $\mathbf{z}_t' = (y_t, \mathbf{x}_t')'$. Assume

$$\begin{bmatrix}\mathbf{z}_t'\\ \mathbf{a}_t'\end{bmatrix} \sim \text{Ind}\left(\begin{bmatrix}\boldsymbol{\mu}\\ \mathbf{0}\end{bmatrix},\begin{bmatrix}\boldsymbol{\Sigma}_{zz} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma}_{aatt}\end{bmatrix}\right),\ t = 1,\ldots,n \tag{3.2}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix}\beta_0 + \boldsymbol{\mu}_x\boldsymbol{\beta}_1\\ \boldsymbol{\mu}_x\end{bmatrix},$$

$$\boldsymbol{\Sigma}_{zz} = \begin{bmatrix}\sigma_{yy} & \boldsymbol{\sigma}_{yx}\\ \boldsymbol{\sigma}_{xy} & \boldsymbol{\Sigma}_{xx}\end{bmatrix},$$

$$\sigma_{yy} = \boldsymbol{\beta}_1'\boldsymbol{\Sigma}_{xx}\boldsymbol{\beta}_1 + \sigma_{qq},$$

and

$$\sigma_{xy} = \Sigma_{xx}\beta_1.$$

In addition, assume that Σ_{xx} is positive definite. Note that

$$\Sigma_{xx}^{-1}\sigma_{xy} = \Sigma_{xx}^{-1}\Sigma_{xx}\beta_1 = \beta_1.$$

Thus, a reasonable estimator of β_1 is

$$\tilde{\beta}_1 = \tilde{\Sigma}_{xx}^{-1}\tilde{\sigma}_{xy}, \tag{3.3}$$

where $\tilde{\Sigma}_{xx}$ and $\tilde{\sigma}_{xy}$ are the corresponding submatrices of the estimated generalized least squares estimator $\tilde{\Sigma}_{zz}$ defined in (2.11).

Theorem 2. Let model (3.1) - (3.2) and the assumptions of Theorem 1 hold. Also assume that Σ_{xx} is positive definite. Then

$$n^{1/2}(\tilde{\beta}_1 - \beta_1) \xrightarrow{L} N(0, \Gamma_{\beta\beta}),$$

where

$$\tilde{\beta}_1 = \tilde{\Sigma}_{xx}^{-1}\tilde{\sigma}_{xy}$$

$$\Gamma_{\beta\beta} = \Sigma_{xx}^{-1}(b'\otimes A)\Phi_p\Gamma\Phi_p'(b\otimes A')\Sigma_{xx}^{-1},$$

$$A_{(p-1)xp} = [0, I_{p-1}],$$

and

$$b_{px1} = [1, -\beta_1']'.$$

Proof. Omitted. □

Thus for large samples, $\tilde{\beta}_1$ is approximately normally distributed. This large sample result can be used to construct confidence intervals and hypothesis tests for β_1 using $\tilde{\beta}_1$.

4. Comparison of Theoretical Variances

Hasabelnaby's estimator (1.6) is generally better than estimator (1.4) when there are heterogeneous error variances. It is of interest to compare our 2-stage method-of-moments estimator $\tilde{\beta}$, defined in (3.3), with Hasabelnaby's estimator $\hat{\beta}$. In this section we compare the variances of the limiting distributions of estimators (1.6) and (3.3) under a specific model. Both estimators are consistent.

Assume that $Z_t = (Y_t, X_t)'$, $t = 1,2,..,n$, are observed. Furthermore assume that the observations came from the model, a special case of (3.1) - (3.2),

$$y_t = \beta_0 + x_t\beta_1 + q_t,$$

$$Z_t = z_t + a_t,$$

and

$$\begin{bmatrix} z_t' \\ a_t' \end{bmatrix} \sim NI\left(\begin{bmatrix} \mu \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{zz} & 0 \\ 0 & \Sigma_{aatt} \end{bmatrix}\right), t = 1,\ldots,n$$

where $\sigma_{qq} = 0.2$, $\beta_1 = 1$, and $\sigma_{xx} = 1$.

Assume that one half of the observations have

$$\Sigma_{aatt} = \begin{bmatrix} .40 & 0 \\ 0 & .10 \end{bmatrix} \equiv \Sigma_{aa11},$$

and the other half of the observations have

$$\Sigma_{aatt} = \begin{bmatrix} .10 & 0 \\ 0 & .40 \end{bmatrix} \equiv \Sigma_{aa22}.$$

It follows from the assumptions that the Z_t are normally distributed. Under this model

$$\sigma_{yy} = \sigma_{xx}\beta_1^2 + \sigma_{qq} = 1.2$$

and

$$\sigma_{xy} = \sigma_{xx}\beta_1 = 1,$$

so

$$\Sigma_{zz} = \begin{bmatrix} 1.2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Hence, for half of the observations,

$$\Sigma_{ZZtt} = \begin{bmatrix} 1.6 & 1 \\ 1 & 1.1 \end{bmatrix},$$

and, for the other half of the observations,

$$\Sigma_{ZZtt} = \begin{bmatrix} 1.3 & 1 \\ 1 & 1.4 \end{bmatrix}.$$

As a result,

$$V_{mm11}^{-1} = \begin{bmatrix} 1.0474 & -1.9044 & 0.8657 \\ -1.9044 & 4.7784 & -2.7701 \\ 0.8657 & -2.7701 & 2.2161 \end{bmatrix},$$

and

$$V_{mm22}^{-1} = \begin{bmatrix} 1.4575 & -2.0821 & 0.7436 \\ -2.0821 & 4.1939 & -1.9334 \\ 0.7436 & -1.9334 & 1.2567 \end{bmatrix}.$$

Let the assumptions of Theorem 2 hold. By Corollary 1.1,

$$\Gamma = \bar{V}_I^{-1}$$

where

$$\bar{\mathbf{V}}_I = \lim_{n\to\infty} n^{-1} \sum_{t=1}^{n} \mathbf{V}_{mmtt}^{-1}.$$

For our model,

$$\bar{\mathbf{V}}_I = .5(\mathbf{V}_{mm11}^{-1} + \mathbf{V}_{mm22}^{-1})$$

$$= \begin{bmatrix} 1.2525 & -1.9933 & 0.8046 \\ -1.9933 & 4.4862 & -2.3517 \\ 0.8064 & -2.3517 & 1.7364 \end{bmatrix}.$$

By Theorem 2, the variance of the limiting distribution of $n^{1/2}(\tilde{\beta}_1 - \beta_1)$ is

$$\Gamma_{\tilde{\beta}\tilde{\beta}} = \sigma_{xx}^{-2}(\mathbf{b}'\otimes\mathbf{A})\Phi_2\Gamma\Phi_2'(\mathbf{b}\otimes\mathbf{A}'),$$

where

$$\mathbf{A} = [0, 1], \ \ \mathbf{b} = [1, -1]',$$

and

$$\Phi_2 = \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix}.$$

Therefore,

$$\Gamma_{\tilde{\beta}\tilde{\beta}} = [0 \ 1 \ -1]\begin{bmatrix} 3.9656 & 2.7539 & 1.8922 \\ 2.7539 & 2.6811 & 2.3551 \\ 1.8922 & 2.3551 & 2.8887 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= 0.860.$$

Recall that

$$\sigma_{vvtt} = \sigma_{qq} + \sigma_{wwtt} - 2\beta_1\sigma_{uwtt} + \beta_1^2\sigma_{uutt}.$$

Therefore for one half of the observations,

$$\sigma_{vvtt} = 0.20 + 0.40 - 0 + 1\cdot(0.10) = 0.70$$

and for the other half of the observations,

$$\sigma_{vvtt} = 0.20 + 0.10 - 0 + 1\cdot(0.40) = 0.70.$$

Hence, the σ_{vvtt} are equal for all observations. In this case, the weighted estimator (1.6) is equivalent to the unweighted estimator (1.4).

The variance of the limiting distribution of $n^{1/2}(\hat{\beta}_1 - \beta_1)$ is

$$\Gamma_{\hat{\beta}\hat{\beta}} = \{\sigma_{vv}\sigma_{xx}^{-1} + \sigma_{xx}^{-2}[.5(\sigma_{uu11} + \sigma_{uu22})\sigma_{vv} + .5(\sigma_{uv11}^2 + \sigma_{uv22}^2)]\},$$

where

$$\sigma_{uvtt} = \sigma_{uwtt} - \sigma_{uutt}\beta_1 = -\sigma_{uutt},$$

for our model. Hence,

$$\Gamma_{\hat{\beta}\hat{\beta}} = \{.70(1) + (1)[.5(.10 + .40).70 + .5(.01 + .16)]\}$$
$$= \{.70 + [(.25).70 + .085]\}$$
$$= 0.960.$$

Therefore, the relative efficiency of estimator $\tilde{\beta}_1$ to Hasabelnaby's estimator $\hat{\beta}_1$ is

$$\mathrm{re}(\tilde{\beta}_1, \hat{\beta}_1) = (0.860)^{-1}(0.960)$$
$$= 1.12.$$

Results from Monte Carlo simulations with this model and these parameters show that for samples as small as size 20, the 2-stage method-of-moments estimator $\tilde{\beta}_1$ is more efficient than Hasabelnaby's estimator $\hat{\beta}_1$. Different choices of $\boldsymbol{\Sigma}_{aa11}$ and $\boldsymbol{\Sigma}_{aa22}$ give different relative efficiencies.

This material is based upon work supported under a National Science Foundation Graduate Fellowship. Support was also provided by the U.S. Bureau of the Census through Joint Statistical Agreement J.S.A. 91-1.

REFERENCES

Amemiya, Yasuo. 1982. Estimators for the errors-in-variables model. Unpublished Ph.D. dissertation, Iowa State University, Ames, Iowa.

Bentler, P.M. 1991. Structural equation models as nonlinear regression models. Paper presented at the International Workshop on Statistical Modelling and Latent Variables, Trento, Italy, July, 1991.

Billingsley, Patrick. 1979. *Probability and Measure.* John Wiley and Sons, New York.

Chung, Kai Lai. 1974. *A Course in Probability Theory.* 2nd ed. Academic Press, New York.

Dahm, P.F. 1979. Estimation of the parameters of the multivariate linear errors in variables model. Unpublished Ph.D. dissertation, Iowa State University, Ames, Iowa.

Dolby, G.R. 1976. The ultrastructural relation: A synthesis of the functional and structural relations. *Biometrika* 63:39–50.

Fuller, Wayne A. 1976. *Introduction to Statistical Time Series.* John Wiley and Sons, New York.

Fuller, Wayne A. 1987. *Measurement Error Models.* John Wiley and Sons, New York.

Fuller, Wayne A. 1990. Prediction of true values for the measurement error model. In P. J. Brown and W. A. Fuller, eds. *Statistical Analysis of Measurement Error Models and Applications.* American Mathematical Society, Providence, Rhode Island.

Graybill, F.A. 1976. *Theory and Application of the Linear Model.* Duxbury Press, North Scituate, Massachusetts.

Hasabelnaby, Nancy A. 1987. The use of a weighting function in measurement error regression. Unpublished Ph.D. dissertation, Iowa State University, Ames, Iowa.

Hoffman, Kenneth, and Ray Kunze. 1971. *Linear Algebra.* 2nd ed. Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Rao, C. R. 1965. *Linear Statistical Inference and Its Applications.* 2nd ed. John Wiley and Sons, New York.

Royden, H. L. 1968. *Real Analysis.* 2nd ed. MacMillan Publishing Co., Inc., New York.