1. Introduction

The statistical consideration of models containing measurement errors began as early as 1877. See Fuller (1987, p. 30). Most of the past work has been done on the univariate linear model with constant error variances. More recently, work has been done on multivariate, non-linear, and non-constant error variance models. We will consider a method of moments estimator for the parameters of a univariate linear model with heteroskedastic error variances.

The general univariate linear measurement error model is

\[ y_t = x_t' \beta + q_t + w_t, \quad t = 1, 2, \ldots, n, \] (1.1)

where \( y_t \) is the dependent observation at time \( t \), \( x_t \) is a 1 x \( k \) vector of explanatory variables, \( \beta \) is a \( k \) x 1 vector of unknown coefficients, and \( q_t \) is the equation error. The usual goal is to estimate \( \beta \). We assume that we are unable to observe \( y_t = (y_t', x_t') \) directly. Instead, we observe \( Z_t = (Y_t', X_t') \), such that

\[ Y_t = y_t' + w_t, \]
\[ X_t = x_t + u_t, \quad t = 1, 2, \ldots, n, \] (1.2)

where \( a_t = (w_t', u_t') \) are random measurement errors.

We further assume that

\[
\begin{bmatrix}
  x_t' \\
  q_t \\
  w_t \\
  u_t'
\end{bmatrix}
\sim NI
\begin{bmatrix}
  \mu_x \\
  0 \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  \Sigma_{xx} & 0 & 0 & 0 \\
  0 & \sigma_{qq} & 0 & 0 \\
  0 & 0 & \sigma_{wtt} & \Sigma_{wtt} \\
  0 & 0 & \Sigma_{uwt} & \Sigma_{utt}
\end{bmatrix}
\begin{bmatrix}
  x_t \\
  q_t \\
  w_t \\
  u_t'
\end{bmatrix}
\]

(1.3)

where

\[
\Sigma_{aatt} = \begin{bmatrix}
  \sigma_{wtt} & \Sigma_{wtt} \\
  \Sigma_{uwt} & \Sigma_{utt}
\end{bmatrix}, t = 1, \ldots, n,
\]

are known for each \( t \). This is the heteroskedastic measurement error model with normal distribution assumptions.

If the \( \Sigma_{aatt} \) are known, a natural estimator is

\[
\hat{\beta} = \hat{M}_{xx}^{-1} \hat{M}_{xy}
\]

(1.4)

where

\[
\hat{M}_{xx} = n^{-1} \sum_{t=1}^{n} (X_t' X_t - \Sigma_{uwt})
\]

and

\[
\hat{M}_{xy} = n^{-1} \sum_{t=1}^{n} (X_t' Y_t - \Sigma_{uwt}).
\]

Fuller (1987, Section 3.1) showed that

\[ \hat{V}^{-1/2} (\hat{\beta} - \beta) \xrightarrow{L} N(0, I), \]

where

\[
\hat{V}_{\beta\beta} = n^{-1} \hat{M}_{xx}^{-1} \hat{G} \hat{M}_{xx}^{-1}
\]

\[
\hat{G} = n^{-1} \sum_{t=1}^{n} (X_t' X_t \hat{\sigma}_{vtt} + \hat{\Sigma}_{uwt} \hat{\Sigma}_{vtt}),
\]

\[
\hat{\sigma}_{vtt} = \hat{\sigma}_{qq} + \sigma_{wtt}^2 \hat{\Sigma}_{uwt} + \hat{\beta}' \Sigma_{uwt} \hat{\beta},
\]

and

\[ \hat{\Sigma}_{uwt} = \hat{\Sigma}_{uwt} - \Sigma_{uwt} \hat{\beta} \hat{\beta}'. \]

The estimator \( \hat{\beta} \) is a consistent estimator and relatively easy to compute. However, it is sometimes possible to construct an asymptotically superior estimator using \( \hat{\beta} \) as a preliminary estimator. Hasabelnaby (1987) investigated the weighted estimator,

\[
\tilde{\beta} = [\sum_{t=1}^{n} \hat{\sigma}_{vtt}^{-1} (X_t' X_t - \Sigma_{uwt})]^{-1}
\times \sum_{t=1}^{n} \hat{\sigma}_{vtt}^{-1} (X_t' Y_t - \Sigma_{uwt}).
\]

(1.6)

where \( \hat{\sigma}_{vtt} \) is an estimator of the variance of \( v_t = q_t + w_t - u_t' \beta \), is defined in (1.5). Hasabelnaby
showed that,
\[ \hat{V}_{\beta \beta}^{-1/2} (\hat{\beta} - \beta) \xrightarrow{L} N(0, I), \]
where
\[ \hat{V}_{\beta \beta} = n^{-2} M_{xx}^{-1} \left( \sum_{t=1}^{n} \sigma_{uvt}^{-1} (X'_t X_t) + \sigma_{vvt}^{-1} \Sigma_{uvt}^{-1} \right) M_{xx}^{-1}. \]
and
\[ \hat{M}_{xx}^{-1} = \sum_{t=1}^{n} \sigma_{vvt}^{-1} (X'_t X_t - \Sigma_{uvt}). \]

The use of \( \sigma_{vvt}^{-1} \) as a weight minimizes the part of the covariance matrix of the limiting distribution associated with \( x'_t v_t \). There is also a contribution of \( u'_t v_t \) to the covariance matrix. The set of weights minimizing the variance of the limiting distribution depends on the unknown \( x'_t \). Since a consistent estimator of \( x'_t \) does not exist, a best set of weights cannot be constructed without additional assumptions.

Under certain assumptions, it is possible to construct an estimator of \( \beta \) that is generally better than (1.6) and better than (1.4). The remainder of this paper will discuss a method of moments estimator based on a two-stage generalized least squares estimator of \( \Sigma_{zz} \) developed by Fuller (1990).

2. Estimation of the Covariance Matrix

Consider the model
\[ Z_t = s_t + a_t, t = 1, ..., n \quad (2.1) \]
where \( Z_t \) is a \( p \)-dimensional row vector of observed values, \( s_t \) is a \( p \)-dimensional row vector of true values, \( a_t \) is a \( p \)-dimensional row vector of measurement errors, and \( n > p \).

Assume that
\[ \left[ s'_t, a'_t \right] \sim NI \left( \left[ \mu, 0 \right], \left[ \Sigma_{zz}, 0 \right], 0, \Sigma_{aatt} \right), t = 1, ..., n \quad (2.2) \]
where \( \Sigma_{aatt} \) is known for all \( t \). Let
\[ \Sigma_{ZZtt} = \Sigma_{zz} + \Sigma_{aatt}. \]
Assume that \( \Sigma_{ZZtt} \) is nonsingular for all \( t \). Define
\[ m_{zz} = (n-1)^{-1} \sum_{t=1}^{n} (Z'_t - \bar{Z})(Z'_t - \bar{Z}), \]
\[ \Sigma_{aa..} = n^{-1} \sum_{t=1}^{n} \Sigma_{aatt}, \]
and
\[ \bar{Z} = n^{-1} \sum_{t=1}^{n} Z_t. \quad (2.3) \]

Fuller (1989) considered the following preliminary estimator of \( \Sigma_{zz} \)
\[ \hat{\Sigma}_{zz} = m_{zz} - \Sigma_{aa..}. \quad (2.4) \]

Fuller also modified (2.4) to ensure that \( \hat{\Sigma}_{zz} \) is positive semidefinite. For simplicity, we will initially assume that estimator (2.4) is positive semidefinite.

Under model (2.1)–(2.2) it can be shown that
\[ E(\hat{\Sigma}_{zz}) = \Sigma_{zz}. \]

Define
\[ \hat{\Sigma}_{ZZtt} = \hat{\Sigma}_{zz} + \Sigma_{aatt}. \quad (2.5) \]
This estimator is an unbiased preliminary estimator of \( \Sigma_{ZZtt} \).

Let \( M \) be any \( p \times p \) matrix. Let vec \( M \) denote the \( 2p \) column vector obtained by listing the columns of \( M \) one beneath the other in a single column. Let vech \( M \) denote the \( p(p+1)/2 \) column vector obtained by listing the elements in each column that are on or below the diagonal. In addition, let \( \Phi_p \) be the \( 2p \times p(p+1)/2 \) matrix such that
\[ \text{vec} M = \Phi_p \text{vech} M. \]

See, for example, Fuller (1987, Appendix 4A) for discussion of vec, vech, and the matrix \( \Phi_p \).

Fuller considered the second round estimators given by
\[ \hat{\mu}' = [ \sum_{t=1}^{n} \hat{W}_{ZZtt}^{-1} \sum_{t=1}^{n} \hat{W}_{ZZtt}^{-1} Z'_t ], \quad (2.6) \]
\[ \text{vech} \hat{m} = n(n-1)^{-1} \left( \sum_{t=1}^{n} \hat{V}_{mmtt}^{-1} \times \sum_{t=1}^{n} \hat{V}_{mmtt}^{-1} \text{vech}[(Z'_t - \hat{\mu})(Z'_t - \hat{\mu})], \quad (2.7) \]
vech $\Sigma_{axa..} = \left( \sum_{t=1}^{n} \hat{\Sigma}_{Zt}^{mm} \right)^{-1} \sum_{t=1}^{n} \hat{\Sigma}_{Zt}^{mm} vech \Sigma_{aatt}.$ 

(2.8)

\[ \hat{\Sigma}_{Zt} = \hat{\Sigma}_{Zt}^{mm} = \left( n^{-1}(n-1) \hat{\Sigma}_{Zt} + n^{-1} m_{ZZ} \right)^{-1}. \]  

(2.10)

The $\hat{\Sigma}_{Zt}$ are preliminary estimators of $\Sigma_{Zt}. $

The inclusion of the $n^{-1} m_{ZZ}$ term in $\hat{\Sigma}_{Zt}$ produces a more stable estimator for $\Sigma_{Zt}. $ Hence, $\hat{\mu}$ is an estimated generalized least squares estimator of $\mu.$ Also, under the normality assumption (2.2),

\[ \hat{\Sigma}_{Zt}^{mm} = \hat{\Sigma}_{Zt}^{mm} = \frac{1}{2} \hat{\Sigma}_{Zt}^{mm} \otimes \frac{1}{2} \hat{\Sigma}_{Zt}^{mm} \mathbf{1}_{p}^{\otimes p}, \]  

(2.9)

is the inverse of the covariance matrix of $\text{vech}[(Z_t - \mu)'(Z_t - \mu)],$

and (2.9) is an estimator of $\Sigma_{mm}^{-1}. $ See Fuller (1987, p. 386). Thus

\[ \hat{\Sigma}_{Zt} = \hat{\Sigma}_{Zt}^{mm} - \Sigma_{axa..} \]  

(2.11)

is the estimated generalized least squares estimator of $\Sigma_{zz}. $ For simplicity we again initially assume that $\hat{\Sigma}_{Zt}^{mm} (2.11) $ is positive semidefinite.

The estimator $\text{vech} \hat{\Sigma}_{Zt}$ is asymptotically normal.

Theorem 1. Assume

1. $Z_t = z_t + a_t, t = 1, \ldots, n,$

2. $\tilde{z}_t = \tilde{a}_t, t = 1, \ldots, n,$

3. $\Sigma_{aatt}$ is known for all $t,$

4. the true value, $\theta_0', $ is in the interior of $\Theta = \{ \theta | \Sigma_{zz} (\theta) \text{ is positive semidefinite} \},$

5. there exists an convex, open set $\Omega \subset \Theta$ with $\theta_0$ in the interior such that $\det(\Sigma_{Zt}(\theta)) \geq c > 0 \forall t$ and $\forall \theta \in \Omega,$

6. $\left[ n^{-1} \sum_{t=1}^{n} \hat{\Sigma}_{Zt}^{mm} (\theta_0) \right]^{-1}$ is uniformly bounded,

(7) $\lim_{n \to \infty} n^{-1} \sum_{t=1}^{n} \hat{\Sigma}_{Zt}^{mm} = \hat{\Sigma}_{Zt},$ where $\hat{\Sigma}_{Zt}$ is positive definite,

(8) $\lim_{n \to \infty} n^{-1} \sum_{t=1}^{n} \hat{\Sigma}_{Zt}^{mm} \mathbf{G}_{mm} = \mathbf{G}_{VV},$

(9) $Z_t$ have uniformly bounded $4+6$ moments.

Then

\[ \n^{-1/2} \text{vech} \hat{\Sigma}_{ZZ} - \Sigma_{zz} \xrightarrow{\mathcal{L}} N(0, \Gamma) \]

where $\Gamma = \mathbf{V}^{-1}_{I} \mathbf{G}_{VV} \mathbf{V}^{-1}_{I}.$

Proof. Omitted. □

Corollary 1.1 Let the assumptions of Theorem 1 hold and, in addition, assume the $Z_t$ to be normally distributed. Then

\[ \Gamma = \hat{\Sigma}_{Zt}^{-1}. \]

Proof. Omitted. □

3. Estimation of Regression Parameters

Consider the following model.

\[ y_t = \beta_0 + x_t \beta_1 + a_t, \]

(3.1)

where $Z_t = (y_t, \mathbf{x}_t)'$ and $Z_t' = (y_t, \mathbf{x}_t)'.$ Assume

\[ \left[ \begin{array}{c} \mathbf{z}_t' \\mathbf{a}_t' \end{array} \right] \sim \text{Ind} \left( \left[ \begin{array}{c} \mu \\Sigma_{zz} \mathbf{0} \\
0 \\Sigma_{aatt} \end{array} \right] \right), t = 1, \ldots, n \]

(3.2)

where

\[ \mu = \left[ \begin{array}{c} \beta_0 + \mu_x \beta_1 \\
\mu_x \end{array} \right], \]

\[ \Sigma_{zz} = \left[ \begin{array}{cc} \sigma_{yy} & \sigma_{yx} \\
\sigma_{xy} & \Sigma_{xx} \end{array} \right], \]

\[ \sigma_{yy} = \beta_1 \Sigma_{xx} \beta_1 + \sigma_{qq} \]

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and 
\[ \sigma_{xy} = \Sigma_{xx} \beta_1. \]
In addition, assume that \( \Sigma_{xx} \) is positive definite.

Note that
\[ \Sigma_{xx}^{-1} \sigma_{xy} = \Sigma_{xx}^{-1} \Sigma_{xx} \beta_1 = \beta_1. \]
Thus, a reasonable estimator of \( \beta_1 \) is
\[ \tilde{\beta}_1 = \Sigma_{xx}^{-1} \sigma_{xy}, \tag{3.3} \]
where \( \Sigma_{xx} \) and \( \sigma_{xy} \) are the corresponding submatrices of the estimated generalized least squares estimator \( \tilde{\Sigma}_{zz} \) defined in (2.11).

Theorem 2. Let model (3.1) - (3.2) and the assumptions of Theorem 1 hold. Also assume that \( \Sigma_{xx} \) is positive definite. Then
\[ n^{1/2}(\tilde{\beta}_1 - \beta_1) \xrightarrow{L} N(0, \Gamma) \]
where
\[ \tilde{\beta}_1 = \Sigma_{xx}^{-1} \sigma_{xy}, \]
\[ \Gamma = \Sigma_{xx}^{-1} \left( b \otimes A \right) \Sigma_{xx}^{-1} \left( b \otimes A' \right) \]
and
\[ b_{p \times 1} = [1, -\beta_1']'. \]

Proof. Omitted. \( \Box \)

Thus for large samples, \( \tilde{\beta}_1 \) is approximately normally distributed. This large sample result can be used to construct confidence intervals and hypothesis tests for \( \beta_1 \) using \( \tilde{\beta}_1 \).

4. Comparison of Theoretical Variances

Hasabelnaby's estimator (1.6) is generally better than estimator (1.4) when there are heterogeneous error variances. It is of interest to compare our 2-stage method-of-moments estimator \( \tilde{\beta} \), defined in (3.3), with Hasabelnaby's estimator \( \hat{\beta} \). In this section we compare the variances of the limiting distributions of estimators (1.6) and (3.3) under a specific model. Both estimators are consistent.

Assume that \( Z_t = (Y_t', X_t')', t = 1, 2, \ldots, n \), are observed. Furthermore assume that the observations came from the model, a special case of (3.1) - (3.2),
\[ y_t = \beta_0 + x_t \beta_1 + q_t, \]
and
\[ Z_t = z_t + a_t, \]
where \( \sigma_{qq} = 0.2, \beta_1 = 1, \) and \( \sigma_{xx} = 1. \)
Assume that one half of the observations have
\[ \Sigma_{aatt} = \begin{bmatrix} .40 & 0 \\ 0 & .10 \end{bmatrix} \equiv \Sigma_{aat1}, \]
and the other half of the observations have
\[ \Sigma_{aatt} = \begin{bmatrix} .10 & 0 \\ 0 & .40 \end{bmatrix} \equiv \Sigma_{aat2}. \]
It follows from the assumptions that the \( Z_t \) are normally distributed. Under this model
\[ \sigma_{yy} = \sigma_{xx} \beta_1^2 + \sigma_{qq} = 1.2 \]
and
\[ \sigma_{xy} = \sigma_{xx} \beta_1 = 1, \]
so
\[ \Sigma_{zz} = \begin{bmatrix} 1.2 & 1 \\ 1 & 1 \end{bmatrix}. \]
Hence, for half of the observations,
\[ \Sigma_{ZZtt} = \begin{bmatrix} 1.6 & 1 \\ 1 & 1 \end{bmatrix}, \]
and, for the other half of the observations,
\[ \Sigma_{ZZtt} = \begin{bmatrix} 1.3 & 1 \\ 1 & 1.4 \end{bmatrix}. \]
As a result,
\[ V_{mm11}^{-1} = \begin{bmatrix} 1.0474 & -1.9044 & 0.8657 \\ -1.9044 & 4.7784 & -2.7701 \\ 0.8657 & -2.7701 & 2.2161 \end{bmatrix}, \]
and
\[ V_{mm22}^{-1} = \begin{bmatrix} 1.4575 & -2.0821 & 0.7436 \\ -2.0821 & 4.1939 & -1.9334 \\ 0.7436 & -1.9334 & 1.2567 \end{bmatrix}. \]

Let the assumptions of Theorem 2 hold. By Corollary 1.1,
\[ \Gamma = V_I^{-1} \]
where
\[ \bar{\mathbf{V}}_1 = \lim_{n \to \infty} \sum_{t=1}^{n} \mathbf{V}^{-1} \mathbf{m} \mathbf{m}^t. \]

For our model,
\[ \bar{\mathbf{V}}_1 = 0.5(\mathbf{V}^{-1}_{11} + \mathbf{V}^{-1}_{22}). \]

By Theorem 2, the variance of the limiting distribution of \( n^{1/2}(\hat{\beta}_1 - \beta_1) \) is
\[ \Gamma_{\beta\beta} = -2(\mathbf{b}' \Theta \mathbf{A})\Phi_2 \Gamma_{\Phi} \Phi_2', \]
where
\[ \mathbf{A} = [0, 1], \quad \mathbf{b} = [1, -1]', \]
and
\[ \Phi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Therefore,
\[ \Gamma_{\beta\beta} = \begin{bmatrix} 3.9656 & 2.7539 & 1.8922 \\ 2.7539 & 2.6811 & 2.3551 \\ 1.8922 & 2.3551 & 2.8887 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0.860. \]

Recall that
\[ \sigma_{vtt} = \sigma_{qq} + \sigma_{wwtt} - 2\beta_1 \sigma_{uwtt} + \beta_1^2 \sigma_{uutt}. \]

Therefore for one half of the observations,
\[ \sigma_{vtt} = 0.20 + 0.40 - 0 + 1 \cdot (0.10) = 0.70 \]
and for the other half of the observations,
\[ \sigma_{vtt} = 0.20 + 0.10 - 0 + 1 \cdot (0.40) = 0.70. \]

Hence, the \( \sigma_{vtt} \) are equal for all observations. In this case, the weighted estimator (1.6) is equivalent to the unweighted estimator (1.4).

The variance of the limiting distribution of \( n^{1/2}(\hat{\beta}_1 - \beta_1) \) is
\[ \Gamma_{\beta\beta} = \{ \sigma_{vv}^{-1} \sigma_{xx} - \frac{5}{4} (\sigma_{uu11} + \sigma_{uu22}) \} \]
where
\[ \sigma_{uu1} = \sigma_{uu2} - \sigma_{uutt}\beta_1 = -\sigma_{uutt}. \]
for our model. Hence,
\[ \Gamma_{\beta\beta} = \{ 0.70(1) + 0.5(1.00 + 0.16) \} = \{ 0.70 + (0.16) \} = 0.960. \]

Therefore, the relative efficiency of estimator \( \hat{\beta}_1 \) to Hasabelnaby's estimator \( \hat{\beta}_1 \) is
\[ \text{re}(\hat{\beta}_1, \hat{\beta}_1) = (0.860)^{-1}(0.960) = 1.12. \]

Results from Monte Carlo simulations with this model and these parameters show that for samples as small as size 20, the 2-stage method-of-moments estimator \( \hat{\beta}_1 \) is more efficient than Hasabelnaby's estimator \( \hat{\beta}_1 \). Different choices of \( \Sigma_{aa11} \) and \( \Sigma_{aa22} \) give different relative efficiencies.

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**REFERENCES**


