

# COST-VARIANCE OPTIMIZATION OF DUAL FRAME DESIGN FOR ESTIMATING PROPORTIONS

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## 1. Introduction

A theoretical framework for multiple frame surveys was first provided by Hartley (1962). A general methodology was developed for utilizing any number of frames without requiring any prior knowledge as to the extent of their mutual overlap. Cochran (1967), Lund (1968), and Fuller and Burmeister (1972) have also studied the problem and offered alternative estimation procedures. Bosecker and Ford (1976) extended the multiple frame estimator developed by Hartley (1962) to take advantage of stratification within the overlap domain for the two frame estimation. As discussed by the above authors, most multiple frame surveys involve only two frames: (i) an area frame, and (ii) a list frame. The list frame is often cheaper and easier to apply, but rarely covers the whole population. The area frame on the other hand would usually cover the whole population, but the unit cost for surveying is generally higher as compared to that for the list frame.

The subpopulation which is covered by the area frame but not the list frame is called the "non-overlap" domain. The remaining part of the population which is covered by both frames, is called the "overlap" domain. By combining the two estimates of the "overlap" domain and adding that to the estimate for the "non-overlap" domain, a single multiple frame estimator is obtained using both the area and list frames.

In this paper, an iterative procedure to obtain the optimum solution for sample allocation and estimation in the context of a dual frame design is given when one of the frames covers the whole population and the study variable is dichotomous, i.e. takes on values 0 or 1. The dichotomous variable is encountered in practice when estimating proportions, e.g. estimation of proportion of population unemployed. Also we have considered a complex sample design for each of the frames, e.g. stratified multistage design. This is accommodated by using design effects as in Kish (1965). The results on efficiency are obtained from an empirical study that uses various combinations of cost ratios, design effect ratios and coverage by the incomplete frame when total survey cost is fixed. The study also uses values of the proportions pertaining to the "overlap" and "non-overlap" domains that are typical of the proportion of population unemployed when the "overlap" domain consists of households with telephones. Contour plots of the efficiencies are also shown. Finally, losses in efficiency due to non-optimum allocation of the sample are given.

## 2. Dual Frame Design

We consider two frames A and B with  $P_A$  and  $P_B$  respectively the population counts for the two frames. Let "a" be the domain defined by the population in frame A only, "b" be the domain defined by the population in frame B only, and "ab" be the domain defined by the population belonging to both frames. The corresponding population counts will be denoted by  $P_a$ ,  $P_b$ , and  $P_{ab}$  respectively. In the special case when frame A covers the whole population, the "overlap" domain  $ab=B$  and  $P_b=0$ , and  $P_A = P_a + P_{ab} = P_a + P_B$ .

For example, in household surveys, the frame A is the area frame and frame B consists of those households which can be contacted by telephone.

Two independent samples  $S_A$  and  $S_B$  with sampling rates  $1/W_A$  and  $1/W_B$  respectively are selected from the frames A and B. The sample design in each of the frames could be complex, e.g. stratified multistage design. The study variable "y" is a dichotomous variable so that it takes on values 0 or 1, i.e. absence or presence of a characteristic, e.g. unemployment. The parameter of interest is the total

$$Y_A = \sum_{i \in A} y_i,$$

where  $\sum_{i \in A}$  is the sum over all the individuals belonging to frame A.

The sample  $S_A$  provides unbiased estimates for the "overlap" and "non-overlap" domains whereas the sample  $S_B$  provides an unbiased estimate for the "overlap" domain only. Following Hartley (1962), the multiple frame estimator of the total  $Y_A$  can be expressed as

$$\hat{Y}_A = \hat{Y}_a + \alpha \hat{Y}_{ab} + (1-\alpha) \hat{Y}_B$$

where

$\hat{Y}_a$  = the area frame estimate of the "non-overlap" domain,

$\hat{Y}_{ab}$  = the area frame estimate of the "overlap" domain,

$\hat{Y}_B$  = the list frame estimate of the "overlap" domain,

and  $\alpha$  is the weight attached to the area frame estimate of the "overlap" domain and has value between 0 and 1. First we note that  $\hat{Y}_A$  is unbiased for  $Y_A$ . This follows from the fact that

$$E(\hat{Y}_{ab}) = Y_B,$$

and

$$Y_A = Y_a + Y_B.$$

The inverse sampling fractions  $w_A, w_B$  and the weight  $\alpha$  are determined so that the variance of  $\hat{Y}_A$  is minimized for fixed total cost. Next we obtain the variance of the estimate  $\hat{Y}_A$  and then using a linear cost model, obtain the optimum values of the above parameters.

### 2.1 Variance Function

The variance of  $\hat{Y}_A$ , say  $V$  is given by

$$V = D_A(W_A-1) \left\{ Y_a \left(1 - \frac{Y_a}{P_A}\right) + \alpha^2 Y_B \left(1 - \frac{Y_B}{P_A}\right) - 2\alpha \frac{Y_a Y_B}{P_A} \right\} + D_B(W_B-1) (1-\alpha)^2 Y_B \left(1 - \frac{Y_B}{P_B}\right),$$

where  $D_A$  and  $D_B$  are the design effects for sample designs for frames A and B respectively.  $Y_B$  is the total of the y variable for the "overlap" domain and  $Y_a$  is the total of the y variable for the "non-overlap" domain. As noted earlier, we have

$$Y_a = Y_A - Y_B.$$

The remaining parameters in the above variance expression are as defined earlier. Ignoring the finite population correction (fpc), the variance  $V$  can be written as:

$$V = D_A W_A \left\{ Y_a \left(1 - \frac{Y_a}{P_A}\right) + \alpha^2 Y_B \left(1 - \frac{Y_B}{P_A}\right) - 2\alpha \frac{Y_a Y_B}{P_A} \right\} + D_B W_B (1-\alpha)^2 Y_B \left(1 - \frac{Y_B}{P_B}\right).$$

## 2.2 Cost Model

Under a linear cost model, the total cost of the dual frame design will be

$$C = C_0 + \frac{P_A C_A}{W_B} + \frac{P_B C_B}{W_B},$$

where  $C_0$  is the fixed cost, and  $C_A$  and  $C_B$  are the per unit costs for frames A and B respectively.

Now we obtain the optimum values of the parameters  $W_A$ ,  $W_B$  and  $\alpha$  using the above variance and cost.

## 2.3 Optimization

The above optimization problem can be solved using one of the following optimization criteria:

- (i) Minimize the total cost  $C$  for required variance  $V$ ,
- (ii) Minimize the variance  $V$  for a given total cost  $C$ ,
- (iii) Minimize the product of the variance  $V$  and the total cost  $C$ .

It can be shown that each of the above 3 criteria gives the same optimum solution for the parameter  $\alpha$  and relative sampling rates in the two frames, i.e.  $W_A/W_B$ .

We will use criteria (iii) and obtain optimum values of  $\alpha$  and the ratio  $W_A/W_B$ . Then the optimum  $W_A$  and  $W_B$  values can be obtained either to satisfy criteria (i) or criteria (ii). This technique was used by Hughes and Rao (1979), and has also been used by Choudhry et al (1985).

First define

$$V^* = \frac{V}{D_B P_A},$$

and

$$C^* = \frac{C - C_0}{C_B P_A}$$

Then the minimization of the product of  $V$  and  $C$  is equivalent to the minimization of the product of  $V^*$  and  $C^*$ .

$V^*$  can be written as

$$V^* = W_A \left\{ \delta \left[ R_a (1-\lambda) (1-R_a + \lambda R_a) + \alpha^2 \lambda R_B \right] - 2\alpha R_a R_B \lambda (1-\lambda) \right\} + \frac{1}{\omega} (1-\alpha)^2 \lambda R_B (1-R_B)$$

where

$$\lambda = P_B/P_A \text{ (relative coverage by the incomplete frame)}$$

$$\delta = D_A/D_B \text{ (design effect ratio)}$$

$$\omega = W_A/W_B \text{ (relative sampling rate in frame B)}$$

Note that a value of  $\omega > 1$  implies that sampling rate in frame B is greater than in frame A.

$$R_a = Y_a/P_a \text{ (proportion of population with the characteristic of interest, e.g. unemployment in the "non-overlap" domain)}$$

$$R_B = Y_B/P_B \text{ (proportion of population with the characteristic of interest in the "overlap" domain).}$$

For the sake of simplicity, we will write  $V^*$  as

$$V^* = W_A \left[ \delta (K_1 + \alpha^2 K_2 - 2\alpha K_3) + \frac{1}{\omega} (1-\alpha)^2 K_4 \right], \quad (1)$$

where

$$K_1 = R_a (1-\lambda) (1-R_a + \lambda R_a),$$

$$K_2 = \lambda R_B (1-\lambda R_B),$$

$$K_3 = R_a R_B \lambda (1-\lambda),$$

$$K_4 = \lambda R_B (1-R_B).$$

$C^*$  can be written as

$$C^* = \frac{1}{W_A} (\eta + \omega \lambda),$$

where

$$\eta = C_A/C_B \text{ (cost ratio between two frames),}$$

and

$\omega$  and  $\lambda$  are as defined above.

We find the optimum values of  $\omega$  and  $\alpha$  such that the product  $V^* C^*$  is minimized.

Let  $f = V^* C^*$

$$= \left( \frac{\eta}{\omega} + \lambda \right) \left[ \delta (\omega K_1 + \omega \alpha^2 K_2 - 2\omega \alpha K_3) + (1-\alpha)^2 K_4 \right]$$

The optimum values of  $\omega$  and  $\alpha$  are obtained by solving the pair of equations obtained by setting the partial derivatives of the function "f" with respect to  $\omega$  and  $\alpha$ , equal to 0, i.e.

$$\frac{\partial f}{\partial \omega} = 0, \text{ and } \frac{\partial f}{\partial \alpha} = 0.$$

$\frac{\partial f}{\partial \omega} = 0$  gives

$$\omega = \left[ \frac{\eta (1-\alpha)^2 K_4}{\lambda \delta (K_1 + \alpha^2 K_2 - 2\alpha K_3)} \right]^{\frac{1}{2}}, \quad (2)$$

and  $\frac{\partial f}{\partial \alpha} = 0$  gives

$$\alpha = \frac{\delta K_3 + K_4/\omega}{\delta K_2 + K_4/\omega}. \quad (3)$$

Thus we have a system of 2 non-linear equations in two unknowns,  $\omega$  and  $\alpha$ . We can solve equations (2) and (3) iteratively to obtain the optimum solution, say  $\omega^*$ ,  $\alpha^*$  as follows:

Step (1): Set  $\omega_0=1$  and obtain  $\alpha_0 = \alpha(\omega_0)$ .

Step (2): At the  $i$ th iteration obtain  $\omega_i = \omega(\alpha_{i-1})$ , and  $\alpha_i = \alpha(\omega_i)$ .

Repeat step (2) for  $i=1, 2, \dots, I$ , where  $I$  is such that

$$|\omega_I - \omega_{I-1}| < \epsilon,$$

and  $|\alpha_I - \alpha_{I-1}| < \epsilon$ .

Then the solution  $\omega_I$  and  $\alpha_I$  is taken as the optimum solution  $\omega^*$ ,  $\alpha^*$ . The parameter  $\epsilon$  is an arbitrary level of tolerance in the error, e.g.  $10^{-4}$ . Note that we have used  $\omega_0 = 1$  as the initial value for the iterative process, any other positive value would converge to the same solution. The  $\omega_0 = 1$  implies that the initial guess is that the two frames have the same sampling rates. From the optimum  $\omega^*$  value, the optimum values of the sampling rates  $1/w_A$  and  $1/w_B$  can be obtained to satisfy the cost or variance constraint.

### 3. Efficiency Relative to Single Frame Design

We define the efficiency as the ratio of the variances for the single frame design and dual frame design when the total cost  $C$  is fixed. The parameters for the single frame design will be denoted with a single slash (') as superscript and those for the dual frame design will be denoted with a double slash (") as superscript.

The single frame design will use only the frame A as it covers the whole population. The variance of the estimate of the total  $Y_A$  obtained from the single frame design (ignoring fpc) is

$$\begin{aligned} V' &= W_A' D_A Y_A \left(1 - \frac{Y_A}{P_A}\right) \\ &= W_A' D_A P_A R_A (1 - R_A) \end{aligned} \quad (4)$$

where

$1/W_A'$  = Sampling rate for single frame design using frame A

$R_A = \frac{Y_A}{P_A}$  (proportion of population with the characteristic of interest in the whole population).

Note that  $R_A$  can be expressed as

$$R_A = (1-\lambda) R_A + \lambda R_B,$$

i.e. the proportion in the whole population is a weighted average of the proportions in the "non-overlap" and "overlap" domains, where the weights are proportional to their respective populations.

The total cost  $C$  can be written as

$$C = C_0' + \frac{P_A C_A}{W_A'} \quad (5)$$

where  $C_0'$  is the fixed cost for single frame design. For the dual frame design, the total cost  $C$  will be written as

$$C = C_0'' + \frac{C_B P_A}{W_A''} (\eta + \omega^* \lambda), \quad (6)$$

where  $C_0''$  is the fixed cost for dual frame design.  $1/W_A''$  is the sampling rate in frame A for dual frame design and  $\omega^*$  is the optimum value of  $\omega$ .

From (5) and (6), we obtain

$$\begin{aligned} \frac{C - C_0'}{C - C_0''} &= \left(\frac{P_A C_A}{W_A'}\right) \left(\frac{W_A''}{C_B P_A (\eta + \omega^* \lambda)}\right) \\ &= \left(\frac{1}{W_A'}\right) \left(\frac{C_A}{C_B}\right) \left(\frac{W_A''}{\eta + \omega^* \lambda}\right) \\ &= \left(\frac{\eta}{W_A'}\right) \left(\frac{W_A''}{\eta + \omega^* \lambda}\right) \end{aligned}$$

or

$$W_A' = \left(\frac{\eta}{k_0}\right) \left(\frac{W_A''}{\eta + \omega^* \lambda}\right) \quad (7)$$

where  $k_0 = (C - C_0') / (C - C_0'')$ .

Thus for fixed total cost  $C$ , the single frame design sampling rate  $1/W_A'$  is  $\left(\frac{k_0}{\eta}\right) \left(\frac{\eta + \omega^* \lambda}{W_A''}\right)$ . In practice the fixed cost for dual frame design will be greater than that for the single frame design, i.e.  $C_0'' > C_0'$  and hence  $k_0 > 1$ . By substituting the value of  $W_A'$  given by (7) into the variance expression for the single frame design as in (4), we obtain

$$V' = \frac{\eta W_A''}{k_0 (\eta + \omega^* \lambda)} D_A P_A R_A (1 - R_A)$$

and the variance for the dual frame design, say  $V''$  from eq. (1) will be

$$\begin{aligned} V'' &= \frac{W_A''}{\omega^*} D_B P_A [\delta (\omega^* K_1 + \omega^* \alpha^{*2} K_2 - 2 \omega^* \alpha^* K_3) \\ &\quad + (1 - \alpha^*)^2 K_4] \end{aligned}$$

The efficiency of the dual frame design relative to the single complete frame design will be defined as the ratio of the above two variances, i.e.

$$\begin{aligned} Eff &= V' / V'' \\ &= \frac{\eta \delta R_A (1 - R_A)}{k_0 \left(\frac{\eta}{\omega^*} + \lambda\right) [\delta (\omega^* K_1 + \omega^* \alpha^{*2} K_2 - 2 \omega^* \alpha^* K_3) + (1 - \alpha^*)^2 K_4]} \end{aligned}$$

In the next section we report some empirical results on the efficiency comparisons. We have computed these efficiencies under the assumption that  $C_0'' = C_0'$ , i.e.

$k_0 = 1$ . The reported values can be adjusted by dividing by appropriate value of  $k_0$ .

#### 4. Empirical Results

We have computed the efficiencies of the dual frame design relative to the single frame design using parameter values encountered in practice when estimating the proportion of population unemployed using an area frame as the complete frame and a telephone frame as an incomplete frame for the underlying population. The telephone frame can be implemented with random digit dialing (RDD) or it can be a list frame of published residential telephone numbers. A list of published residential telephone numbers can be obtained from the directory listing or directly from the telephone company.

The per unit cost in the area frame is generally higher than that for the list frame due to high travel cost involved in implementing an area frame sample design. Similarly the design effect for the area frame sample design is higher than that for the list frame sample design due to more clustering in the area frame sample design. The proportion of population unemployed is also likely to be higher in the "non-overlap" domain (i.e. households without telephones) compared to that in the "overlap" domain (i.e. household with telephones), for example see Drew et. al (1988). The coverage by the incomplete frame (i.e. households with telephones) is usually high and is in the 80%-90% range. Therefore we have considered the following parameter values for the purpose of empirical evaluation of the dual frame sample design.

Cost Ratio ( $\eta$ ) = 1.5, 2.0, 3.0

Design Effect Ratio ( $\delta$ ) = 1.2, 1.5

$R_a/R_B = 1.5, 2.0, 3.0$ , where  $R_a$  and  $R_B$  are proportions of population unemployed in the "non-overlap" and "overlap" domains respectively.

The relative efficiencies have been obtained for the above parameter values when the coverage by the telephone frame ( $P_B/P_A$ ) is 0.80 and the proportion unemployed in the telephone frame (i.e.  $R_B$ ) is 0.05 and 0.08 (Tables 1 and 2). The efficiencies were also

obtained for ( $P_B/P_A$ ) value equal to 0.90 when  $R_B$  is 0.05 and 0.08 and all other parameter values are the same as above (Tables 3 and 4). Besides efficiencies, the corresponding optimum values of the relative sampling rates in the telephone frame ( $w^*$ ) and the weights ( $\alpha^*$ ) attached to the estimates of "overlap" domain obtained from area frame are also reported. From these tables, the following observations are made.

- (1) As expected, the efficiency of the dual frame design increases both with the cost ratio and the design effect ratio.
- (2) As the cost ratio increases, the relative sampling rate for the telephone frame increases and consequently the weight attached to the area frame estimate of the "overlap" domain decreases. The impact due to increase in the design effect ratio is similar to that due to increase in cost ratio.
- (3) As the proportion of unemployed in the "non-overlap" domain increases relative to that in the "overlap" domain, the sampling rate in the telephone frame decreases and consequently the weight attached to area frame estimate of the "overlap" domain increases. Moreover, the efficiency gains also decrease. The optimum allocation and the corresponding efficiency depend only on the relative proportions in the "non-overlap" and "overlap" domains (i.e.  $R_a/R_B$ ) and not on the magnitude of these proportions.
- (4) As the coverage of the telephone frame ( $P_B/P_A$ ) increases, the relative sampling rate in the telephone frame also increases and consequently the weight attached to the area frame estimate of the "overlap" domain decreases. Moreover the efficiency gains also increase.

The efficiency contour plots in Figure 1 show the combination of cost ratio ( $\eta$ ) and design effect ratio ( $\delta$ ) values for which similar efficiency gains are realized when the coverage by telephone frame ( $\lambda = P_B/P_A$ ) is 0.80,  $R_B = 0.05$ , and  $R_a/R_B = 2.0$ .

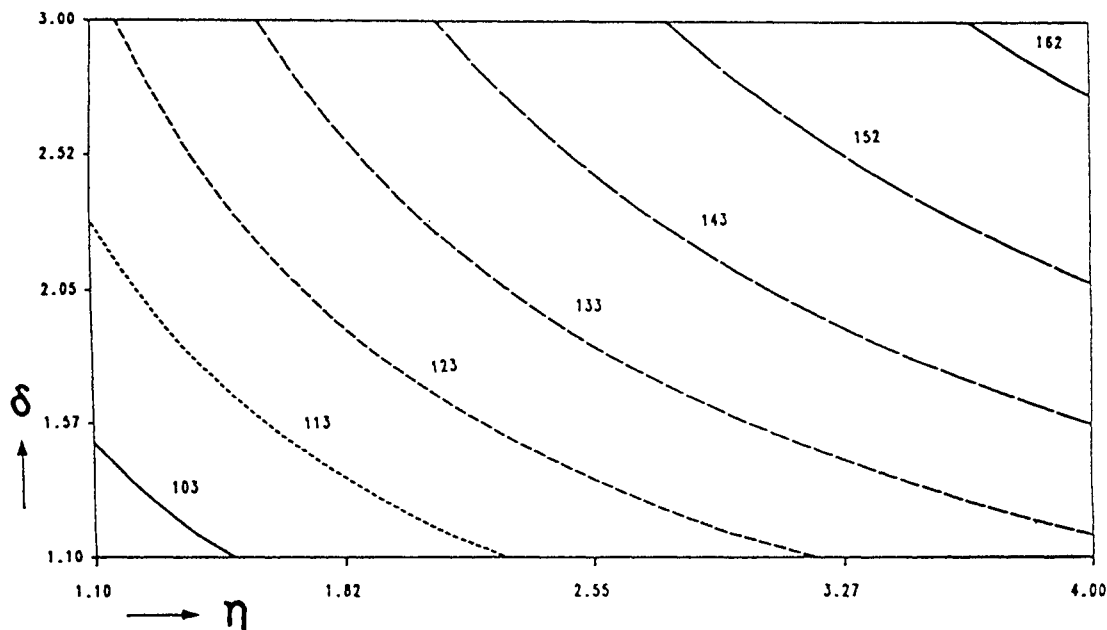


Fig. 1: Efficiency Contour Plots ( $\lambda=0.80, R_B = 0.05, R_a/R_B=2.0$ )

We also investigated the impact on the efficiency of a non-optimum allocation of the sample. We note from table 3 that the relative efficiency of optimum dual frame sample design is 122% when cost ratio ( $\eta$ ) and design effect ratio ( $\delta$ ) are both equal to 1.5, the coverage by the telephone frame ( $P_B/P_A$ ) is equal to 0.90, the proportion of unemployed in the "overlap" domain ( $R_B$ ) is 0.05 and that in the "non-overlap" domain is twice as large. The corresponding optimum value of  $\omega$  is 1.02, but it was found that for values of  $\omega$  in the range 0.50-1.75, the efficiency drops to a minimum of only 118% from its optimum value of 122%. Thus the optimum design is very robust against misspecification of sampling rates.

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**References**

Bosecker, R.R., and Ford, B.L. (1976), "Multiple Frame Estimation with Stratified Overlap Domain," Proceedings of the Social Statistics Section, ASA, pp. 219-224.  
 Choudhry, G.H., Lee, H., and Drew, J.D. (1985), "Cost-Variance Optimization for the Canadian Labour Force Survey," Survey Methodology, Vol. 11, pp. 33-50.

Cochran, R.S. (1967), "The Estimation of Domain Sizes when Sampling Frames are Interlocking," Proceedings of the Social Statistics Section, ASA, pp. 332-335.  
 Drew, J.D., Choudhry, G.H., and Hunter, L.A. (1988), "Nonresponse Issues in Government Telephone Surveys," in TELEPHONE SURVEY METHODOLOGY, Edited by R.M. Groves, P.P. Biemer, L.E. Lyberg, J.T. Massey, W.L. Nicholls, and J. Waksberg, John Wiley and Sons, New York.  
 Fuller, W.A., and Burmeister, L.F. (1972), "Estimators for Samples Selected from Two Overlapping Frames," Proceedings of the Social Statistics Section, ASA, pp. 245-249.  
 Hartley, H.O. (1962), "Multiple Frame Surveys," Proceedings of the Social Statistics Section, ASA, pp. 203-206.  
 Hughes, E., and Rao, J.N.K. (1979), "Some Problems of Optimal Allocation in Sample Surveys involving Inequality Constraints," Commun. Statist.-Theor. Meth., A8 (15), pp. 1551-1574, Marcel Dekker, Inc.  
 Kish, L. (1965), "Survey Sampling," John Wiley and Sons, New York.  
 Lund, E.R. (1968), "Estimators in Multiple Frame Surveys," Proceedings of the Social Statistics Section, ASA, pp. 282-288.

Table 1: Percent Efficiency ( $V^j/V^n$ ) for  $\lambda = 0.80, R_B = 0.05$

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	0.66	0.56	109	1.04	0.45	118	1.64	0.34	133
	2.0	0.45	0.65	105	0.79	0.52	112	1.30	0.40	124
	3.0	0.22	0.80	101	0.49	0.64	105	0.91	0.49	113
1.5	1.5	0.76	0.47	115	1.09	0.39	126	1.61	0.30	142
	2.0	0.57	0.55	110	0.85	0.45	118	1.30	0.35	131
	3.0	0.34	0.67	105	0.57	0.55	110	0.93	0.43	119

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

Table 2: Percent Efficiency ( $V^j/V^n$ ) for  $\lambda = 0.80, R_B = 0.08$

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	0.63	0.57	108	1.01	0.46	117	1.60	0.36	132
	2.0	0.44	0.66	105	0.76	0.54	111	1.26	0.41	123
	3.0	0.20	0.81	101	0.46	0.66	105	0.87	0.51	113
1.5	1.5	0.74	0.48	115	1.06	0.40	125	1.57	0.31	141
	2.0	0.55	0.56	110	0.82	0.46	117	1.26	0.36	130
	3.0	0.32	0.69	104	0.54	0.57	109	0.90	0.45	118

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

Table 3: Percent Efficiency ( $V^j/V^n$ ) for  $\lambda = 0.90, R_B = 0.05$

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	1.16	0.42	117	1.73	0.33	132	2.61	0.25	158
	2.0	0.89	0.49	112	1.39	0.38	125	2.14	0.29	146
	3.0	0.57	0.60	107	0.98	0.47	116	1.59	0.35	132
1.5	1.5	1.28	0.35	129	1.76	0.28	146	2.51	0.22	174
	2.0	1.02	0.40	122	1.43	0.32	136	2.08	0.25	160
	3.0	0.71	0.49	113	1.04	0.40	124	1.57	0.31	141

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

Table 4: Percent Efficiency ( $V^j/V^n$ ) for  $\lambda = 0.90, R_B = 0.08$

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	1.13	0.43	117	1.69	0.34	132	2.55	0.25	157
	2.0	0.86	0.50	112	1.35	0.39	124	2.09	0.30	145
	3.0	0.55	0.61	106	0.94	0.48	115	1.54	0.37	130
1.5	1.5	1.25	0.35	128	1.72	0.29	145	2.45	0.22	173
	2.0	0.99	0.41	121	1.40	0.33	135	2.03	0.26	158
	3.0	0.68	0.50	113	1.01	0.41	123	1.53	0.32	140

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

**Table 1: Percent Efficiency ( $V'/V''$ ) for  $\lambda = 0.80, R_B = 0.05$**

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	0.66	0.56	109	1.04	0.45	118	1.64	0.34	133
	2.0	0.45	0.65	105	0.79	0.52	112	1.30	0.40	124
	3.0	0.22	0.80	101	0.49	0.64	105	0.91	0.49	113
1.5	1.5	0.76	0.47	115	1.09	0.39	126	1.61	0.30	142
	2.0	0.57	0.55	110	0.85	0.45	118	1.30	0.35	131
	3.0	0.34	0.67	105	0.57	0.55	110	0.93	0.43	119

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

**Table 2: Percent Efficiency ( $V'/V''$ ) for  $\lambda = 0.80, R_B = 0.08$**

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	0.63	0.57	108	1.01	0.46	117	1.60	0.36	132
	2.0	0.44	0.66	105	0.76	0.54	111	1.26	0.41	123
	3.0	0.20	0.81	101	0.46	0.66	105	0.87	0.51	113
1.5	1.5	0.74	0.48	115	1.06	0.40	125	1.57	0.31	141
	2.0	0.55	0.56	110	0.82	0.46	117	1.26	0.36	130
	3.0	0.32	0.69	104	0.54	0.57	109	0.90	0.45	118

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

**Table 3: Percent Efficiency ( $V'/V''$ ) for  $\lambda = 0.90, R_B = 0.05$**

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	1.16	0.42	117	1.73	0.33	132	2.61	0.25	158
	2.0	0.89	0.49	112	1.39	0.38	125	2.14	0.29	146
	3.0	0.57	0.60	107	0.98	0.47	116	1.59	0.35	132
1.5	1.5	1.28	0.35	129	1.76	0.28	146	2.51	0.22	174
	2.0	1.02	0.40	122	1.43	0.32	136	2.08	0.25	160
	3.0	0.71	0.49	113	1.04	0.40	124	1.57	0.31	141

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$

**Table 4: Percent Efficiency ( $V'/V''$ ) for  $\lambda = 0.90, R_B = 0.08$**

$\delta$	$R_a/R_B$	$\eta = 1.5$			$\eta = 2.0$			$\eta = 3.0$		
		$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff	$\omega^*$	$\alpha^*$	Eff
1.2	1.5	1.13	0.43	117	1.69	0.34	132	2.55	0.25	157
	2.0	0.86	0.50	112	1.35	0.39	124	2.09	0.30	145
	3.0	0.55	0.61	106	0.94	0.48	115	1.54	0.37	130
1.5	1.5	1.25	0.35	128	1.72	0.29	145	2.45	0.22	173
	2.0	0.99	0.41	121	1.40	0.33	135	2.03	0.26	158
	3.0	0.68	0.50	113	1.01	0.41	123	1.53	0.32	140

$$\delta = D_A/D_B, \quad \eta = C_A/C_B, \quad \lambda = P_B/P_A, \quad \omega = \frac{W_A}{W_B}$$