PROPERTIES OF TWO NEWLY-DEFINED DISTRIBUTIONS AND THEIR RELATIONSHIPS TO OTHER DISTRIBUTIONS

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Abstract

Two newly-defined distributions, sine distribution and cosine distribution, have been proposed in this article. Their basic properties including moments, cumulants, skewness, kurtosis, mean deviation and all kinds of generating functions are discussed. Besides, their relationships to other distributions are also presented including proofs. It is seen that cosine distribution can serve as a very rough approximation to the standard normal distribution under suitable condition of parameter chosen.

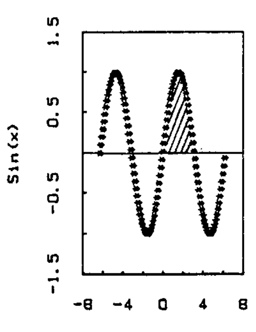
KEY WORDS :

Sine distribution; Cosine distribution; Transformations of random variables.

1. Introduction

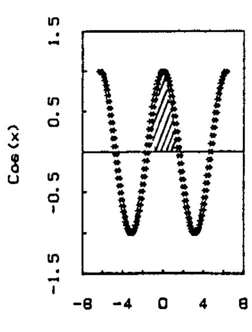
It is well known that the pictures of sine and cosine function look like the following figures.

Figure 1



SINE FUNCTION

Figure 2



COSINE FUNCTION

If we integrate the shaded part of both figures, we'll have

$$\int_0^{\pi} \sin x dx = -\cos x \,|_0^{\pi} = 2 \qquad (1.1)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx = \sin x \,|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \qquad (1.2)$$

Therefore if we let $f(x) = \frac{1}{2}\sin x$ and $g(x) = \frac{1}{2}\cos x$, then both f(x) and g(x) can be treated as probability density functions (abbrev. pdf hereafter) under suitable ranges. Burrows (1986) discussed the extreme statistics from g(x) which he called the sine distribution. Expanding these basic ideas I have defined two new distributions with f(x) and g(x) as their special cases. The main goal of this article is to introduce these new distributions and their basic properties. Also, their relationships to other distributions will be discussed and will play a very crucial role in this article. A very rough normal approximation has been proposed and its comparison to Chew's approximation (Chew 1968) is also given. The names of these distributions are Sine distribution and Cosine distribution due to their corresponding functional forms of pdfs. Finally, a diagram, due to Leemis (1986), showing relationships to others for these two distributions is presented in the appendix.

2. Definitions

2.1 Sine distribution

If random variable X has the following pdf then we say that it has the sine distribution and denote it by X ~ S(x,m).

$$f(x) = s(x;m) = \frac{m}{2}\sin(mx) \quad \text{if } 0 \le x \le \frac{\pi}{m},\ m>0$$
$$= 0 \quad \text{otherwise.} \qquad (2.1)$$

2.2 Cosine distribution

If random variable Y has the following pdf then we say that it has the cosine distribution and denote it by Y ~ Cos(y;m).

$$g(y) = \cos(y;m) = \frac{m}{2}\cos(my) \quad \text{if } |y| \le \frac{\pi}{2m},\ m>0$$
$$= 0 \quad \text{otherwise.} \qquad (2.2)$$

It is no doubt that expressions (2.1) and (2.2) are two pdfs since one can easily verify this just by integrating them over their corresponding ranges. Because the trigonometric functions are periodic and satisfy

$$f(x) = f(x+2n\pi) \quad n \in Z, \qquad (2.3)$$

we can expand those previous two definitions as follows.

2.3 Displaced sine distribution

If random variable X has the following pdf then we say that it has the displaced sine distribution and denote it by X ~ Sd(x;m,n).

$$f(x) = sd(x;m,n) = \frac{m}{2}\sin(mx) \quad \text{if } 0 \le x - \frac{2n}{m}\pi \le \frac{\pi}{m},\ m>0,\ n \in Z$$
$$= 0 \quad \text{otherwise.} \qquad (2.4)$$

2.4 Displaced cosine distribution

If random variable Y has the following pdf then we say that it has the displaced cosine distribution and denote it by Y ~ Cd(y;m,n).

$$g(y) = cd(y;m,n) = \frac{m}{2}\cos(my) \quad \text{if } |y - \frac{2n}{m}\pi| \le \frac{\pi}{2m},\ m>0,\ n \in Z$$
$$= 0 \quad \text{otherwise.} \qquad (2.5)$$

In this article we will discuss only those properties of the sine and cosine distributions since the other two distributions can be obtained just by the linear transformation of these two distributions. Besides, all their properties are almost the same, therefore, it is unnecessary to have a discussion here.

3. Basic properties

3.1 Relationships among moments

Let $\mu'_r = E(X^r)$ and $\mu_r = E(X-\mu)^r$. Then we have the following two lemmas concerning the moment's relationship within these new distributions.

Lemma 3.1.1 If X ~ S(x;m) then all moments of X exist and satisfy the following relation :

$$\mu'_r = \frac{1}{2}\left(\frac{\pi}{m}\right)^r - \frac{r(r-1)}{m^2}\mu'_{r-2} \qquad r=1, 2, 3, ... \qquad (3.1.1)$$

In equation (3.1.1) we have $\mu'_{-1} = 0$ and $\mu'_0 = 1$.

pf:

$$\mu'_r = E(X^r) = \frac{m}{2}\int_0^{\frac{\pi}{m}} x^r \sin(mx) dx \qquad let\ u = x^r,\ dv = \sin(mx) dx$$

$$= \frac{1}{2}\left(\frac{\pi}{m}\right)^r + \frac{r}{2}\int_0^{\frac{\pi}{m}} x^{r-1}\cos(mx)\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{m}\right)^r + \frac{r}{2}\left[\frac{x^{r-1}}{m}\sin(mx)\Big|_0^{\frac{\pi}{m}} - \int_0^{\frac{\pi}{m}} \frac{r-1}{m}x^{r-2}\sin(mx)\,dx\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{m}\right)^r - \frac{r(r-1)}{m^2}\cdot\frac{m}{2}\int_0^{\frac{\pi}{m}} x^{r-2}\sin(mx)\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{m}\right)^r - \frac{r(r-1)}{m^2}\mu'_{r-2}$$

Qed.

Lemma 3.1.2 If X ~ Cos(x;m) then all moments of X exist and satisfy

$$\mu'_r = 0 \qquad \text{if } r \text{ is odd}$$

$$\mu'_r = \left(\frac{\pi}{2m}\right)^r - \frac{r(r-1)}{m^2}\mu'_{r-2} \qquad \text{if } r \text{ is even}$$

$$or \quad \mu'_{2k} = \left(\frac{\pi}{2m}\right)^{2k} - \frac{2k(2k-1)}{m^2}\mu'_{2k-2} \qquad k=1,2,3,\ldots \tag{3.1.2}$$

pf :

$$\mu'_r = E(X^r) = \frac{m}{2}\int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} x^r\cos(mx)\,dx \qquad let\ u = x^r,\ dv = \cos(mx)\,dx$$

$$= \frac{m}{2}\left[\frac{x^r}{m}\sin(mx)\Big|_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} - \int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} \frac{r}{m}x^{r-1}\sin(mx)\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2m}\right)^r + \left(-\frac{\pi}{2m}\right)^r\right] - \frac{r}{2}\int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} x^{r-1}\sin(mx)\,dx$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2m}\right)^r + \left(-\frac{\pi}{2m}\right)^r\right] - \frac{r}{2}\left[-\frac{x^{r-1}}{m}\cos(mx)\Big|_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} + \int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} \frac{r-1}{m}x^{r-2}\cos(mx)\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2m}\right)^r + \left(-\frac{\pi}{2m}\right)^r\right] - \frac{r(r-1)}{m^2}\cdot\frac{m}{2}\int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} x^{r-2}\cos(mx)\,dx \tag{3.1.3}$$

When r = 2k-1 is odd, the right-hand side of equation (3.1.3) is 0, so $\mu'_{2k-1} = 0$. When r = 2k is even, then equation (3.1.3) is reduced to equation (3.1.2) which is the desired result.

Qed.

Two other well known general results about moments and cumulants are obtained from Johnson & Kots (1969). They will be used to find the first four (central) moments and cumulants of these new distributions.

Lemma 3.1.3 Let X be any random variable with finite moments, then

$$\mu_r = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu_1'^{\,j}\mu'_{r-j}, \tag{3.1.4}$$

$$or\quad \mu'_r = \sum_{j=0}^{r}\binom{r}{j}\mu_1'^{\,j}\mu_{r-j}. \tag{3.1.5}$$

From equation (3.1.4) after plugging in r = 2,3,4 respectively, we obtain

$$\mu_2 = \mu'_2 - \mu_1'^2,$$
$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3,$$
$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4. \tag{3.1.6}$$

Lemma 3.1.4 Let X be any random variable with finite moments, then

$$\kappa_1 = \mu'_1 = E(X),$$
$$\kappa_2 = \mu_2 = Var(X),$$
$$\kappa_3 = \mu_3,$$
$$\kappa_4 = \mu_4 - 3\mu_2^2. \tag{3.1.7}$$

3.2 The first four moments and cumulants

By using formula (3.1.1) we can obtain μ'_r, r = 1, 2, 3, 4, for the sine distribution. Then by using formula (3.1.6) we can obtain μ_r and by using formula (3.1.7) we can obtain κ_r for r = 1, 2, 3, 4, for the same distribution. Similarly, we can get μ'_r, μ_r & κ_r for the cosine distribution. Their corresponding first four moments and cumulants are exhibited in table 1 and table 2, respectively.

From tables 1 and 2 we see that μ_2, μ_3 and μ_4 of sine and cosine distribution are the same, as a consequence, their κ_2, κ_3, and κ_4 values are also the same.

3.3 Skewness, kurtosis and the mean deviation

Since sine and cosine distributions have the same values of μ_2, μ_3 and μ_4, they will

Table 1 The first four moments and cumulants of the sine distribution

r	1	2	3	4
μ'_r	$\frac{\pi}{2m}$	$\frac{\pi^2-4}{2m^2}$	$\frac{\pi^3-6\pi}{2m^3}$	$\frac{\pi^4-12\pi^2+48}{2m^4}$
μ_r	0	$\frac{\pi^2-8}{4m^2}$	0	$\frac{\pi^4-48\pi^2+384}{16m^4}$
κ_r	$\frac{\pi}{2m}$	$\frac{\pi^2-8}{4m^2}$	0	$\frac{-\pi^4+96}{8m^4}$

Table 2 The first four moments and cumulants of the cosine distribution

r	1	2	3	4
μ'_r	0	$\frac{\pi^2-8}{4m^2}$	0	$\frac{\pi^4-48\pi^2+384}{16m^4}$
μ_r	0	$\frac{\pi^2-8}{4m^2}$	0	$\frac{\pi^4-48\pi^2+384}{16m^4}$
κ_r	0	$\frac{\pi^2-8}{4m^2}$	0	$\frac{-\pi^4+96}{8m^4}$

also have the same values of the coefficient of skewness α_3 and the kurtosis α_4. The corresponding definitions for α_3 and α_4 are

$$\alpha_3 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3}{\sigma^3}, \tag{3.3.1}$$

$$\alpha_4 = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4}{\sigma^4}. \tag{3.3.2}$$

Therefore, from table 1 and 2 we have

$$\alpha_3 = 0, \tag{3.3.3}$$

$$\alpha_4 = \frac{\pi^4-48\pi^2+384}{\pi^4-16\pi^2+64} = 2.19 \tag{3.3.4}$$

for both distributions.

One can expect that their mean deviations must also be the same. If X ~ S(x;m) then

$$v_1 = E[\,|X-E(X)|\,] = E[\,|X-\frac{\pi}{2m}|\,]$$

$$= \frac{m}{2}\int_0^{\frac{\pi}{m}} |x-\frac{\pi}{2m}|\sin(mx)\,dx$$

$$= \frac{m}{2}\int_0^{\frac{\pi}{2m}}\left(\frac{\pi}{2m}-x\right)\sin(mx)\,dx + \frac{m}{2}\int_{\frac{\pi}{2m}}^{\frac{\pi}{m}}\left(x-\frac{\pi}{2m}\right)\sin(mx)\,dx$$

$$= \frac{m}{2}\int_0^{\frac{\pi}{2m}}\left(\frac{\pi}{2m}-x\right)\sin(mx)\,dx - \frac{m}{2}\int_0^{\frac{\pi}{2m}}\left(x-\frac{\pi}{2m}\right)\sin(mx)\,dx$$

$$= m\int_0^{\frac{\pi}{2m}}\frac{\pi}{2m}\sin(mx)\,dx - m\int_0^{\frac{\pi}{2m}} x\sin(mx)\,dx$$

$$= -\frac{\pi}{2m}\cos(mx)\Big|_0^{\frac{\pi}{2m}} - m\left[-\frac{x}{m}\cos(mx)\Big|_0^{\frac{\pi}{2m}} + \int_0^{\frac{\pi}{2m}}\frac{1}{m}\cos(mx)\,dx\right]$$

$$= \frac{\pi}{2m} - \frac{1}{m} = \frac{\pi-2}{2m} \qquad (3.3.5)$$

Similarly, if Y ~ Cos(y;m) then

$$v_1 = E[\,|Y-E(Y)|\,] = E[\,|Y|\,]$$

$$= \frac{m}{2}\int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} |y| \cos(my)dy$$

$$= m\int_0^{\frac{\pi}{2m}} y\cos(my)dy$$

$$= m\left[\frac{y}{m}\sin(my)\Big|_0^{\frac{\pi}{2m}} - \int_0^{\frac{\pi}{2m}} \frac{1}{m}\sin(my)dy\right]$$

$$= \frac{\pi}{2m} + \frac{1}{m}\cos(my)\Big|_0^{\frac{\pi}{2m}} = \frac{\pi}{2m} - \frac{1}{m} = \frac{\pi-2}{2m} \qquad (3.3.6)$$

In addition, we get an extra result from equation (3.3.5) and (3.3.6), that the ratio of their mean deviation to their standard deviation is also the same and is independent of the parameter m, i.e., is a constant :

$$\frac{v_1}{\sigma} = \frac{\pi-2}{\sqrt{\pi^2-8}} = 0.835\,. \qquad (3.3.7)$$

3.4 Characteristic and generating functions

Lemma 3.4.1 If X ~ S(x;m) then its moment generating function $m_X(t)$, cumulant generating function $\psi_X(t)$ and the characteristic function $\phi_X(t)$ are

$$m_X(t) = \frac{m^2(e^{\frac{\pi}{m}t}+1)}{2(m^2+t^2)} = \frac{m^2}{m^2+t^2}e^{\frac{\pi}{2m}t}\cdot\cosh(\frac{\pi}{2m}t) \qquad (3.4.1)$$

$$\psi_X(t) = \log[m_X(t)] = \log\left[\frac{m^2(e^{\frac{\pi}{m}t}+1)}{2(m^2+t^2)}\right] \qquad (3.4.2)$$

$$\phi_X(t) = m_X(it) = \frac{m^2}{m^2-t^2}e^{i\frac{\pi}{2m}t}\cdot\cosh(i\frac{\pi}{2m}t) \qquad (3.4.3)$$

respectively.

pf : We only need to prove that equation (3.4.1) holds.

$$m_X(t) = E(e^{tX}) = \frac{m}{2}\int_0^{\frac{\pi}{m}} e^{tx}\sin(mx)dx$$

$$= \frac{m}{2}\left[-\frac{1}{m}e^{tx}\cos(mx)\Big|_0^{\frac{\pi}{m}} + \int_0^{\frac{\pi}{m}}\frac{t}{m}e^{tx}\cos(mx)dx\right]$$

$$= \frac{1}{2}(e^{\frac{\pi}{m}t}+1) + \frac{t}{2}\int_0^{\frac{\pi}{m}}\cos(mx)dx$$

$$= \frac{1}{2}(e^{\frac{\pi}{m}t}+1) + \frac{t}{2}\left[\frac{e^{tx}}{m}\sin(mx)\Big|_0^{\frac{\pi}{m}} - \int_0^{\frac{\pi}{m}}\frac{t}{m}e^{tx}\sin(mx)dx\right]$$

$$= \frac{1}{2}(e^{\frac{\pi}{m}t}+1) - \frac{t^2}{2m}\int_0^{\frac{\pi}{m}} e^{tx}\sin(mx)dx$$

$$= \frac{1}{2}(e^{\frac{\pi}{m}t}+1) - \frac{t^2}{m^2}m_X(t)$$

$$\rightarrow m_X(t)(1+\frac{t^2}{m^2}) = \frac{1}{2}(e^{\frac{\pi}{m}t}+1)$$

$$\rightarrow m_X(t) = \frac{m^2(e^{\frac{\pi}{m}t}+1)}{2(m^2+t^2)}$$

Qed.

For the cosine distribution case, we have very similar results. So, I'll just list the formulas for these functions without any proofs.

Lemma 3.4.2 If Y ~ Cos(y;m) then

$$m_Y(t) = \frac{m^2(e^{\frac{\pi}{2m}t}+e^{-\frac{\pi}{2m}t})}{2(m^2+t^2)} = \frac{m^2}{m^2+t^2}\cosh(\frac{\pi}{2m}t) \qquad (3.4.4)$$

$$\psi_Y(t) = \log[m_Y(t)] = \log\left[\frac{m^2}{m^2+t^2}\cosh(\frac{\pi}{2m}t)\right] \qquad (3.4.5)$$

$$\phi_Y(t) = m_Y(it) = \frac{m^2}{m^2-t^2}\cosh(i\frac{\pi}{2m}t) \qquad (3.4.6)$$

4. Relationships to other distributions

This section contains the most important topic of this article and its goal is to demonstrate how these two distributions relate to others. For convenience, they will be discussed separately and proofs of the relationships will be given only for the sine distribution case.

4.1 Sine distribution case

All relationships are based on the transformation of random variables . That is, if X is a random variable with pdf f(x) and Y is another random variable obtained from X by the transformation Y = h(X) = AX, then pdf of Y is given by

$$g(y) = f(A^{-1}y)|J| = f(A^{-1}y)\frac{1}{|\det A|}\,, \qquad (4.1.1)$$

where J is the Jacobian of the transformation.

The first three relationships are fairly easy to see, so their proofs will be skipped here.

R4.1.1 If X ~ S(x;m) then Y = X - $\frac{\pi}{2m}$ ~ Cos(y;m).

R4.1.2 If X ~ S(x;m) then Y = X - $\frac{2n}{m}\pi$ ~ Sd(y;m,n). In fact, the sine distribution is a special case of the displaced sine distribution, i.e. S(x;m) ≡ Sd(x;m,0).

R4.1.3 If X ~ S(x;m) then Y = -cos(mX) ~ U(-1,1) which is uniformly distributed over the interval [-1,1].

R4.1.4 If X ~ S(x;m) then random variable Y = tan[-$\frac{\pi}{2}$cos(mX)] is distributed as the standard Cauchy distribution C(0,1) with pdf

$$g(y) = \frac{1}{\pi(1+y^2)} \qquad -\infty < y < \infty \qquad (4.1.2)$$

pf : X ~ S(x;m) then f(x) = $\frac{m}{2}$sin(mx), $0 \le x \le \frac{\pi}{m}$. Since

$$Y = \tan[-\frac{\pi}{2}\cos(mX)] \rightarrow y = \tan[-\frac{\pi}{2}\cos(mx)]$$

$$\rightarrow dy = \sec^2[-\frac{\pi}{2}\cos(mx)]\frac{m\pi}{2}\sin(mx)dx$$

$$\rightarrow |J| = |\frac{dx}{dy}| = \frac{1}{\sec^2[-\frac{\pi}{2}\cos(mx)]\frac{m\pi}{2}\sin(mx)}$$

$$g(y) = f(x)|J| = \frac{m}{2}\sin(mx)\frac{1}{\sec^2[-\frac{\pi}{2}\cos(mx)]\frac{m\pi}{2}\sin(mx)}$$

$$= \frac{1}{\pi\cdot\sec^2[-\frac{\pi}{2}\cos(mx)]} = \frac{1}{\pi}\cdot\frac{1}{1+\tan^2[-\frac{\pi}{2}\cos(mx)]}$$

$$= \frac{1}{\pi}\frac{1}{1+y^2} = \frac{1}{\pi(1+y^2)}$$

Qed.

R4.1.5 If X ~ S(x;m) then random variable Y = tan[-tan$^{-1}\lambda$·cos(mX)] is distributed as the doubly truncated Cauchy distribution at λ with pdf

$$g(y) = \frac{1}{2\tan^{-1}\lambda(1+y^2)} \qquad -\lambda < y < \lambda \qquad (4.1.3)$$

pf : y = tan[-tan$^{-1}\lambda$cos(mx)] → |J| = $|\frac{dx}{dy}| = |\frac{dy}{dx}|^{-1}$

$$\rightarrow |J| = \frac{1}{\sec^2[-\tan^{-1}\lambda\cos(mx)]m\tan^{-1}\lambda\sin(mx)}$$

$$g(y) = f(x)|J| = \frac{m}{2}\sin(mx)\frac{1}{\sec^2[-\tan^{-1}\lambda\cos(mx)]m\tan^{-1}\lambda\sin(mx)}$$

$$= \frac{1}{2\tan^{-1}\lambda}\cdot\frac{1}{1+\tan^2[-\tan^{-1}\lambda\, cos(mx)]}$$

$$= \frac{1}{2\tan^{-1}\lambda(1+y^2)} \qquad \text{Qed.}$$

R4.1.6 If X ~ S(x;m) then random variable

$$Y = \alpha + 2\beta\tanh^{-1}[-cos(mX)] \tag{4.1.4}$$

is distributed as the logistic distribution with pdf

$$g(y;\alpha,\beta) = \frac{\exp(-\frac{y-\alpha}{\beta})}{\beta[1+exp(-\frac{y-\alpha}{\beta})]^2}$$

$$= \frac{1}{4\beta} sech^2(\frac{y-\alpha}{2\beta}) \quad -\infty < y < \infty,\ -\infty < \alpha < \infty,\ \beta > 0. \tag{4.1.5}$$

pf : $y = \alpha + 2\beta\tanh^{-1}[-cos(mx)]$

$$\rightarrow \tanh(\frac{y-\alpha}{2\beta}) = -cos(mx)$$

$$\rightarrow 1-\cos^2(mx) = 1-\tanh^2(\frac{y-\alpha}{2\beta}) = sech^2(\frac{y-\alpha}{2\beta})$$

$$\rightarrow |J| = |\frac{dx}{dy}| = \frac{1-\cos^2(mx)}{2\beta m\sin(mx)}$$

$$g(y) = f(x)|J| = \frac{m}{2}sin(mx)\frac{1-\cos^2(mx)}{2\beta m\sin(mx)}$$

$$= \frac{1}{4\beta}[1-\cos^2(mx)] = \frac{1}{4\beta}sech^2(\frac{y-\alpha}{2\beta}) \qquad \text{Qed.}$$

R4.1.7 If X ~ S(x;m) then random variable

$$Y = \sinh^{-1}[\tan(-\frac{\pi}{2}\cos(mX))] \tag{4.1.6}$$

is distributed as the hyperbolic secant distribution with pdf

$$g(y) = \frac{1}{\pi}sech(y) \qquad -\infty < y < \infty \tag{4.1.7}$$

pf :

$$y = \sinh^{-1}[\tan(-\frac{\pi}{2}\cos(mx))]$$

$$\rightarrow x = \frac{1}{m}\cos[-\frac{2}{\pi}\tan^{-1}(sinh(y))]$$

$$\rightarrow |J| = |\frac{dx}{dy}| = \frac{|-\frac{2}{\pi}\cdot\frac{cosh y}{1+\sinh^2 y}|}{m\cdot\sqrt{1-[-\frac{2}{\pi}\tan^{-1}(sinh(y))]^2}}$$

$$= \frac{2sech(y)}{m\pi\sin(mx)}$$

$$g(y) = f(x)|J| = \frac{m}{2}sin(mx)\frac{2sech(y)}{m\pi sin(mx)} = \frac{1}{\pi}sech(y). \qquad \text{Qed.}$$

R4.1.8 If X ~ S(x;m) then random variable

$$Y = \exp[\sinh^{-1}[\tan(-\frac{\pi}{2}\cos(mX))]] \tag{4.1.8}$$

is distributed as the half-Cauchy distribution with pdf

$$g(y) = \frac{2}{\pi(1+y^2)} \qquad y > 0 \tag{4.1.9}$$

pf : We can use two different ways to prove this result, one is to use the transformation directly as those given in this section. The other way is to use the previous result R4.1.7 via the hyperbolic secant distribution. Here I give the second method of the proof.

Let $Z = \sinh^{-1}[\tan(-\frac{\pi}{2}cos(mX))]$ from result R4.1.7 and equation (4.1.8) we know that Y = exp(Z) and Z has a hyperbolic secant distribution. Henceforth

$$f(z) = \frac{1}{\pi}sech(z) \qquad -\infty < z < +\infty$$

Now we have

$$y = exp(z) \rightarrow z = ln(y)$$

$$\rightarrow |J| = |\frac{dz}{dy}| = \frac{1}{y}$$

$$\rightarrow g(y) = f(z)|J| = f(ln(y))|J| = \frac{2}{\pi}\frac{1}{y+y^{-1}}\frac{1}{y} = \frac{2}{\pi(1+y^2)} \qquad \text{Qed.}$$

R4.1.9 If X ~ S(x;m) then random variable

$$Y = sin[-\frac{\pi}{2}cos(mX)] \tag{4.1.10}$$

is distributed as the arcsine distribution with pdf

$$g(y) = \frac{1}{\pi\sqrt{1-y^2}} \qquad -1 < y < 1 \tag{4.1.11}$$

pf :

$$y = sin[-\frac{\pi}{2}cos(mx)] \rightarrow x = \frac{2}{\pi}\cos^{-1}(-\sin^{-1}y)$$

$$|J| = |\frac{dx}{dy}| = \frac{1}{m}\frac{\frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}}{\sqrt{1-(-\frac{2}{\pi}\sin^{-1}y)^2}} = \frac{2}{m\pi sin(mx)\sqrt{1-y^2}}$$

Hence

$$g(y) = f(x)|J| = \frac{m}{2}sin(mx)\frac{2}{m\pi sin(mx)\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}} \qquad \text{Qed.}$$

R4.1.10 If X ~ N(0,1) then random variable

$$Y = \frac{\pi}{2m} + \frac{1}{m}\sin^{-1}[2sgn(X)\cdot I(|X|)] \tag{4.1.12}$$

is distributed as the sine distribution. Where

$$I(z) = \int_0^z \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt \tag{4.1.13}$$

is the integral from zero to z under the standard normal curve

$$sgn(x) = +1 \quad \text{if } x \ge 0,$$
$$= -1 \quad \text{if } x < 0. \tag{4.1.14}$$

pf : Let $Z = \frac{1}{m}\sin^{-1}[2sgn(X)\cdot I(|X|)]$ then $Y = Z + \frac{\pi}{2m}$. From result R4.1.1, we need only to prove that Z is cosine distributed, then the result follows.

Since

$$z = \frac{1}{m}\sin^{-1}[2sgn(x)\cdot I(|x|)]$$

therefore, when $x \ge 0$ we have

$$z = \frac{1}{m}\sin^{-1}[2I(x)].$$

Also,

$$I(x) = \int_0^x \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$$

then

$$I'(x) = \frac{\partial}{\partial x}\int_0^x \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = \phi(x)$$

$$dz = \frac{1}{m}\frac{2I'(x)}{\sqrt{1-(2I(x))^2}}dx = \frac{2\phi(x)}{m\cdot cos(mz)}dx$$

$$|J| = |\frac{dx}{dz}| = \frac{m\cdot cos(mz)}{2\phi(x)}$$

Hence

$$g(z) = f(x)|J| = \phi(x)\frac{m\cdot cos(mz)}{2\phi(x)} = \frac{m}{2}cos(mz) = cos(z;m).$$

Similarly, since I(-x) = -I(x), we have, when x < 0

$$z = \frac{1}{m}\sin^{-1}[-2I(-x)] = \frac{1}{m}\sin^{-1}[2I(x)],$$

and the result follows after the same transformation is applied. Qed.

R4.1.11 If $Y = \frac{1}{m}\sin^{-1}[sgn(T)\cdot I_{\frac{T^2}{\nu+T^2}}(\frac{1}{2},\frac{\nu}{2})] + \frac{\pi}{2m}$ is distributed as the sine distribution S(y;m), then $T \sim t_\nu$ is t distributed with ν d.f., where

$$I_x(p,q) = \frac{1}{B(p,q)}\int_0^x y^{p-1}(1-y)^{q-1}dy \tag{4.1.15}$$

is the incomplete beta function.

pf : Let $Z = Y - \frac{\pi}{2m} = \frac{1}{m}\sin^{-1}[sgn(T)\cdot I_{\frac{t^2}{\nu+T^2}}(\frac{1}{2},\frac{\nu}{2})]$ then by result R4.1.1 we have Z ~ Cos(z;m) is cosine distributed.

Hence

$$z=\frac{1}{m}\sin^{-1}\left[sgn(t)\cdot I_{\frac{t^2}{v+t^2}}(\frac{1}{2},\frac{v}{2})\right]$$

$$=\frac{1}{m}\sin^{-1}\left[I_{\frac{t^2}{v+t^2}}(\frac{1}{2},\frac{v}{2})\right] \quad \text{if } t\ge 0$$

$$=\frac{1}{m}\sin^{-1}\left[-I_{\frac{t^2}{v+t^2}}(\frac{1}{2},\frac{v}{2})\right] \quad \text{if } t<0.$$

Since

$$I_{\frac{t^2}{v+t^2}}(\frac{1}{2},\frac{v}{2})=\frac{1}{B(\frac{1}{2},\frac{v}{2})}\int_0^{\frac{t^2}{v+t^2}} y^{-\frac{1}{2}}(1-y)^{\frac{v}{2}-1}dy \tag{4.1.16}$$

we have

$$\frac{\partial}{\partial t}\left[I_{\frac{t^2}{v+t^2}}(\frac{1}{2},\frac{v}{2})\right]=\frac{1}{B(\frac{1}{2},\frac{v}{2})}\frac{\partial}{\partial t}\int_0^{\frac{t^2}{v+t^2}} y^{-\frac{1}{2}}(1-y)^{\frac{v}{2}-1}dy$$

$$=\frac{1}{B(\frac{1}{2},\frac{v}{2})}(\frac{t^2}{v+t^2})^{-\frac{1}{2}}(1-\frac{t^2}{v+t^2})^{\frac{v}{2}-1}\frac{2vt}{(v+t^2)^2}$$

$$=\frac{2}{\sqrt{v}B(\frac{1}{2},\frac{v}{2})}(1+\frac{t^2}{v})^{-\frac{v+1}{2}}$$

Therefore

$$g(t)=f(z)|J|$$

$$=\frac{m}{2}cos(mz)\frac{1}{m}\frac{\frac{2}{\sqrt{v}B(\frac{1}{2},\frac{v}{2})}(1+\frac{t^2}{v})^{-\frac{v+1}{2}}}{\sqrt{1-[I_{\frac{t^2}{v+t^2}}(\frac{1}{2},\frac{v}{2})]^2}}$$

$$=\frac{2}{\sqrt{v}B(\frac{1}{2},\frac{v}{2})}(1+\frac{t^2}{v})^{-\frac{v+1}{2}}=t_v(t)$$

Qed.

4.2 Cosine distribution case

Since we have result R4.1.1, the proof of those similar results will not be given here. The interested reader can prove all of them by using those transformations directly.

R4.2.1 If X ~ Cos(x;m) then -X ~ Cos(x;m). That is, X and -X have the same distribution.

pf : Since cos(x;m) = $\frac{m}{2}cos(mx)$ is an even function, the result follows.

R4.2.2 If X ~ Cos(x;m) then Y = X + $\frac{\pi}{2m}$ ~ S(y;m).

R4.2.3 If X ~ Cos(x;m) then Y = X + $\frac{2n}{m}\pi$ ~ Cd(y;m,n). In fact, we have Cos(x;m) = Cd(x;m,0).

R4.2.4 If X ~ Cos(x;m) then Y = sin(mX) ~ U(-1, 1).

R4.2.5 If X ~ Cos(x;m) then Y = tan[$\frac{\pi}{2}$sin(mX)] ~ C(0, 1).

R4.2.6 If X ~ Cos(x;m) then random variable Y = $tan[\tan^{-1}\lambda sin(mX)]$ is distributed as the doubly truncated Couchy distribution at λ with pdf (4.1.3).

R4.2.7 If X ~ Cos(x;m) then random variable Y = $\alpha+2\beta\tanh^{-1}[sin(mX)]$ is distributed as the logistic distribution with pdf (4.1.5).

R4.2.8 If X ~ Cos(x;m) then random variable Y = $\sinh^{-1}[tan(\frac{\pi}{2}sin(mX))]$ is distributed as the hyperbolic secant distribution with pdf (4.1.7).

R4.2.9 If X ~ Cos(x;m) then random variable Y = $exp[\sinh^{-1}[tan(\frac{\pi}{2}sin(mX))]]$ is distributed as the half-Cauchy distribution with pdf (4.1.9).

R4.2.10 If X ~ Cos(x;m) then random variable Y = $sin[\frac{\pi}{2}sin(mX)]$ is distributed as the arcsine distribution with pdf (4.1.11).

R4.2.11 If X ~ N(0,1) then Y = $\frac{1}{m}\sin^{-1}[2sgn(X)\cdot I(|X|)]$ ~ Cos(y;m), where I(·) and sgn(·) are defined as in equation (4.1.13) and (4.1.14), respectively.

R4.2.12 If T ~ t_v, then Y = $\frac{1}{m}\sin^{-1}\left[sgn(T)\cdot I_{\frac{T^2}{v+T^2}}(\frac{1}{2},\frac{v}{2})\right]$ ~ Cos(y;m).

Finally, since cosine distribution is symmetric about its mean 0, we can use it as a very rough approximation to the standard normal distribution N(0,1) after suitably choosing the value of the parameter m. To let the standard deviation be unity, we need

$$\frac{\pi^2-8}{4m^2}=1. \tag{4.2.1}$$

Hence

$$m=\frac{\sqrt{\pi^2-8}}{2}\approx 0.6836. \tag{4.2.2}$$

Table 3 gives the table of standard normal and two kinds of cosine distributions so that we can see how good the approximation is compared to the real standard normal distribution and the cosine distribution defined by Chew (1968).

Table 3 table of the standard normal and cosine distributions

x*	Normal	Cos1	Cos2
0.0	.5000	.5000	.5000
0.2	.5793	.5720	.5682
0.4	.6554	.6442	.6350
0.6	.7257	.7088	.6994
0.8	.7881	.7702	.7600
1.0	.8413	.8252	.8158
1.2	.8849	.8728	.8657
1.4	.9192	.9122	.9088
1.6	.9452	.9436	.9422
1.8	.9641	.9670	.9714
2.0	.9772	.9832	.9897
2.2	.9861	.9931	.9989
2.4	.9918	.9982	1.0000
2.6	.9953	.9998	
2.8	.9974	1.0000	
3.0	.9987		

* x is multiple of the standard deviation

The Cos1 distribution in the table is given by Chew (1968) with pdf

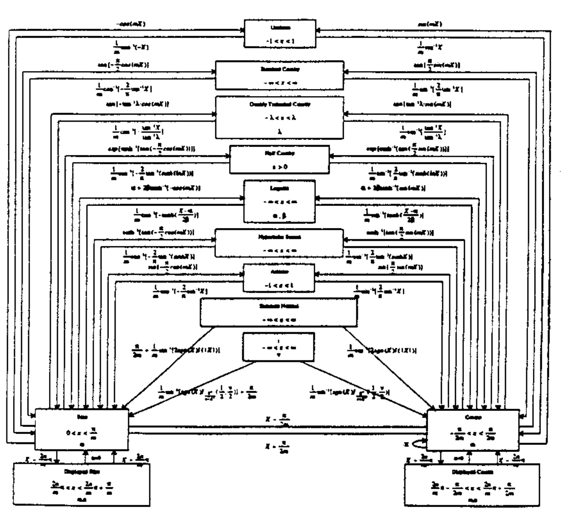
$$f(x)=\frac{1+cos\ x}{2\pi} \quad -\pi\le x\le\pi \tag{4.2.3}$$

while the Cos2 distribution is defined in this article by m = 0.6836, i.e., with pdf

$$f(x)=0.3418\ cos(0.6836x) \quad -2.2978\le x\le 2.2978 \tag{4.2.4}$$

The valid range for my cosine distribution is ±2.2978 and within this range only 98.92% is covered under a standard normal curve. Besides, when a value exceeds 2.3 standard deviations then my approximation will no longer be valid. This is the reason why I say that it is a very rough approximation.

APPENDIX



How to Relate Distributions?

Theorem Suppose random variable X has cdf F(x) and random variable Y has cdf G(y). Suppose further that F(.) and G(.) are continuous with inverse functions exist. If we equate F(x) and G(y) and solve for y in terms of x, we'll have $y = G^{-1}(F(x))$. Then the transformation defined by

$$Y = h(X) = G^{-1}(F(X)) \tag{1}$$

or the inverse transformation

$$X = h^{-1}(Y) = F^{-1}(G(Y)) \tag{2}$$

is the relationship between these two random variables.

Ex.1 Let X ~ C(0,1) then we have

$$F(x) = .5 + (1/\pi)\tan^{-1}x \qquad -\infty < x < \infty . \tag{3}$$

Let Y ~ U(-π/2,π/2) then

$$G(y) = .5 + y/\pi \qquad -\pi/2 < y < \pi/2. \tag{4}$$

Equating F(x) and G(y), we have

$$.5 + (1/\pi)\tan^{-1}x = .5 + y/\pi$$

or

$$y = \tan^{-1}x.$$

Therefore, relationship between the uniform and the standard Cauchy distribution is $Y = \tan^{-1}X$ or $X = \tan Y$ where X and Y have the standard Cauchy distribution C(0,1) and the uniforn distribution over the interval [-π/2,π/2], U(-π/2,π/2), respectively.

Ex.2 Let X ~ Cos(x;m) then its cdf is

$$F(x) = .5 + .5\sin(mx) \qquad -\pi/2m < x < \pi/2m. \tag{5}$$

Let Y be arcsine distributed with cdf

$$G(y) = .5 + (1/\pi)\sin^{-1}y \qquad -1 < y < 1. \tag{6}$$

Then by letting

$$.5 + .5\sin(mx) = .5 + (1/\pi)\sin^{-1}y$$

we have

$$y = \sin((\pi/2)\sin(mx)) \tag{7}$$

or

$$x = (1/m)\sin^{-1}((2/\pi)\sin^{-1}y). \tag{8}$$

Therefore, the relationship between random variable X and Y is

$$Y = \sin((\pi/2)\sin(mX)). \tag{9}$$

ps: see result R4.2.10 .

Ex.3 Let X ~ S(x;m) then its cdf is

$$F(x) = .5 - .5\cos(mx) \qquad 0 < x < \pi/m. \tag{10}$$

Let Y be logistic distributed with cdf

$$G(y) = .5 + .5\tanh((y-\alpha)/2\beta) \qquad -\infty < y < \infty. \tag{11}$$

Equating F(x) and G(y) we have

$$.5 - .5\cos(mx) = .5 + .5\tanh((y-\alpha)/2\beta)$$

then

$$y = \alpha + 2\beta\tanh^{-1}(-\cos(mx)) \tag{12}$$

or

$$x = (1/m)\cos^{-1}(-\tanh((y-\alpha)/2\beta)). \tag{13}$$

Therefore, the relationship between random variable X and Y is

$$Y = \alpha + 2\beta\tanh^{-1}(-\cos(mX)). \tag{4.1.4}$$

ps: see result R4.1.6 .

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