LASPEYRES PRICE INDEX ESTIMATION UNDER AN AUTOREGRESSIVE MODEL

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1. INTRODUCTION

The problem of price index estimation can be approached either as one of economics or of statistics. There is a rich economic literature on the subject with a recent collection of papers being Diewert and Montmarquette (1983). One of the basic economic formulations of the problem is how to find bounds or approximations to the true cost-of-living index defined as the ratio of the minimum cost to a consumer of achieving a certain standard of living when faced with current period prices to the minimum cost of achieving the same standard of living when faced with prices at some base period (Diewert 1983). Various choices of indexes have been studied including Laspeyres, Paasche, and many others.

Given a decision on the type of index that best measures the desired economic concept and that is feasible to compute, one statistical problem is how to best estimate the index based on sample data. In contrast to the economic literature, published studies on the statistical properties of index estimators are relatively few. McCarthy (1961) discussed sampling variability in the U.S. Consumer Price Index (CPI) and how it might be estimated. Kott (1984) reported model-based analyses of some index estimators. Kersten (1985) studied the effect of nonresponse bias in a household budget survey in the Netherlands when the survey was used to construct expenditure weights for a consumer price index. In a related article Balk and Kersten (1986) examined the contribution to the variance of an estimated index due to the sampling variability of the budget survey. Leaver, Weber, Cohen, and Archer (1986) discussed the use of mathematical programming techniques in sample allocation for the U.S. CPI.

This article studies statistical properties of two estimators of a fixed base Laspeyres price index and extends earlier research reported by Valliant (1987b). Sections 2 and 3 introduce notation and define the population index and its estimators. Section 4 presents theoretical properties of the estimators of long and short-term price change. Properties under an assumed model are emphasized with some probability sampling properties being briefly mentioned. The results of a large simulation study using price data from the U.S. CPI are summarized in Section 5. Section 6 concludes.

2. NOTATION AND MODEL

The population of N items is organized into H strata where stratum h contains N_h items. A stratum consists of items such as beef or milk products which are fairly homogeneous in terms of price change. At time t the price of item i in stratum h is p_{hi}^{t} and the price relative between times t and m is $r_{hi}^{t,m} = p_{hi}^{t}/p_{hi}^{m}$. The quantity of item hi purchased in the base period, time 0, is q_{hi}^{0}. The finite population value of the long-term fixed-base Laspeyres price index for comparing period t to period 0 is then defined as

$$I^{t,0} = \frac{\Sigma_h^H \Sigma_i^{N_h} \, p_{hi}^{t} q_{hi}^{0}}{\Sigma_h^H \Sigma_i^{N_h} \, p_{hi}^{0} q_{hi}^{0}}$$

$$= \Sigma_h \Sigma_i \, W_{hi}^{0} r_{hi}^{t,0} \qquad (1)$$

where $W_{hi}^{0} = p_{hi}^{0} q_{hi}^{0} / \Sigma_h \Sigma_j^{N_h} \, p_{hj}^{0} q_{hj}^{0}$. Based on the long-term index, the population short-term index for comparing periods t and m (m<t) is defined as $I^{t,m} = I^{t,0}/I^{m,0}$. Short-term changes that are of particular interest are monthly, quarterly, semiannual, and annual changes. For a variety of economic and operational reasons, including a continually changing universe of items, indexes published by the U.S. government are not intended to be pure fixed base Laspeyres indexes. However, in this paper attention is restricted to the pure form (1).

We will study the properties of various index estimators under the following, simple stationary autoregressive model

$$\begin{aligned} r_{hi}^{t,0} &= \alpha_{th} + \varepsilon_{thi} \\ \varepsilon_{thi} &= \rho_h \varepsilon_{t-1,hi} + u_{thi} \end{aligned} \tag{2}$$

where $E(u_{thi}) = E(u_{thi}u_{t'h'i'}) = 0$, and $E(u_{thi}^2) = \sigma_{uh}^2$. The model allows average price change to differ for different time periods, but, for a given time period, items within a stratum have a common mean. Under (2) we have $\varepsilon_{thi} = \Sigma_{k=0}^{\infty} \rho_h^k u_{t-k,hi}$ (e.g. see Chow 1983, p.79). From this identity and the assumed properties of u_{thi} it follows that the covariance structure implied by model (2) is

$$\mathrm{cov}(r_{hi}^{t,0}, r_{h'i'}^{t-m,0}) = \begin{cases} \Delta_h^2 & m=0, h=h', i=i' \\ \rho_h^m \Delta_h^2 & m\neq 0, h=h', i=i' \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $\Delta_h^2 = \sigma_{uh}^2/(1-\rho_h^2)$. The model thus implies that (a) for a given item, the long-term relatives for different time periods are correlated and (b) for different items, relatives are uncorrelated regardless of time period. The model specifically implies that $\mathrm{corr}(r_{hi}^{t,0}, r_{hi}^{t-1,0}) = \rho_h$ and in most situations this correlation will be fairly large and positive.

An extension of model (2) would be to the possibility of intratemporal correlations between long-term relatives for different items, either in the same stratum or in different strata. Common price determining factors, such as central supply and distribution of some types of items and local economic conditions which affect price movements of many items, may result in intratemporal correlations which are large enough to require consideration in developing a realistic model. As we will illustrate in the remainder of the paper, model (2) is realistic enough to achieve our major aim which is to distinguish important differences in some alternative index estimators.

3. ALTERNATIVE INDEX ESTIMATORS

This section introduces several estimators of long and short-term change which are derived from ones used in U.S. government index programs and from model-based considerations. At each time period τ ($\tau=1,\ldots,t$) we have a sample $s_{\tau h}$ of n_h items from stratum h. We assume that the sample size n_h is the same in all periods.

Over time the sampling proceeds as follows. A sample is selected at time 1 and an attempt is made to follow the same set of sample items in all subsequent time periods. However, because of discontinuation of items, refusals by business establishments to continue participation, establishments going out of business, or for other reasons, sample attrition does occur so that samples from different time periods will only partially overlap. The sample size in each stratum is maintained by substitution of items whose characteristics may be comparable to those of the original sample items or which may be dissimilar to some degree. The particular substitution procedures that are used in several Federal index programs are described in U.S. Bureau of Labor Statistics (BLS) (1982,1984). Another important reason for sample differences between time periods is seasonal items. Items like gardening equipment and certain fresh fruits and vegetables are often available for pricing only in a subset of the months in a year in many parts of the country. In some index programs the sample design itself forces turnover to occur. For example, in the U.S. CPI approximately 20% of the establishments in the sample are rotated out and replaced with different units each year in order to limit the length of time a respondent is in the sample.

We assume that all items, originals, substitutes, and other replacements, follow model (2). A key practical restriction in the index problem is that full information on all items in the universe is not available and that, as a result, the weight W_{hi}^{0} is unknown for each nonsample item. We will assume that only the weights for individual sample items and the aggregate stratum weights $W_h^0 = \Sigma_i W_{hi}^0$ are known (or estimated) and are available for construction of an index estimator.

An estimator of $I^{t,0}$ which is similar to that used in the U.S. CPI, the Producer Price Index, and the International Price Program conducted by BLS is

$$\hat{I}_1^{t,0} = \Sigma_h W_h^0 \Pi_{\tau=1}^{t} \hat{R}_h^{\tau,\tau-1}$$

where $\hat{R}_h^{\tau,\tau-1} = \bar{r}_{\tau h}^{\tau}/\bar{r}_{\tau h}^{\tau-1}$ and $\bar{r}_{\tau h}^{k} = \Sigma_{s_{\tau h}} r_{hi}^{k,0}/n_h$ for $k=\tau$ or $\tau-1$. The estimator $\hat{I}_1^{t,0}$ will be referred to as a product estimator since it is the product of estimated short-term relatives. Both the numerator and denominator of the short-term estimator $\hat{R}_h^{\tau,\tau-1}$ are based on the units in the sample for period τ. As a practical matter this requires an item to be priced in two consecutive periods before it can be used in the index estimator. If the samples are identical for $\tau=1,\ldots,t$ then $\hat{I}_1^{t,0}$ reduces to a simpler estimator

$$\hat{I}_2^{t,0} = \Sigma_h W_h^0 \bar{r}_{th}^{t} .$$

Each long-term estimator can be used to construct short-term estimators in the obvious way : $\hat{I}_j^{t,m} = \hat{I}_j^{t,0}/\hat{I}_j^{m,0}$ for $m<t$, $j=1,2$.

4. THEORETICAL PROPERTIES OF ESTIMATORS

The model-based approach is applied in this section to compare theoretical properties, particularly variances, of the long and short-term estimators. In the following we consider model-variances of the form $var(\hat{I})$ rather than prediction-variances defined as $var(\hat{I}-I)$. For many types of estimators and models the two are asymptotically equivalent (see, e.g. Royall and Cumberland 1978, Valliant 1987a). Some design-based properties of the estimators are also briefly mentioned.

4.1 Estimators of Long-term Price Change

The simple estimator $\hat{I}_2^{t,0}$ is design-unbiased under a sampling plan in which items are selected with probabilities equal to $n_h W_{hi}^0/W_h^0$. Under that type of plan $\hat{I}_1^{t,0}$ is an approximately design-unbiased estimator of (1) when the sample in each stratum is large. In practice a new probability-proportionate-to-size (pps) sample is not selected for each time period, and the sample at a particular time is the result of additions, deletions, and substitutions made to a sample selected at an earlier period. When the population itself has missing prices for some items, due to seasonality, absence of transactions in a period, or for other reasons, then the expectations of $\hat{I}_1^{t,0}$ and $\hat{I}_2^{t,0}$ in pps sampling can be shown to be

$$E_p(\hat{I}_1^{t,0}) \doteq \Sigma_h W_h^0 (\tilde{W}_{th}^0/\tilde{W}_{0h}^0) I_h^{t,0} \text{ and}$$

$$E_p(\hat{I}_2^{t,0}) = \Sigma_h W_h^0 I_h^{t,0} \qquad (4)$$

where E_p denotes expectation with respect to the sample design, $I_h^{t,0} = \Sigma_h \Sigma_{i \in C_{th}} W_{hi}^0 r_{hi}^{t,0} / \tilde{W}_t^0$, $\tilde{W}_{jh}^0 = \Sigma_{i \in C_{jh}} W_{hi}^0$ $(j=0,t)$, $\tilde{W}_t^0 = \Sigma_h \tilde{W}_{th}^0$, and C_{jh} is the set of items in stratum h with nonmissing prices at time j $(j=0,t)$. If $\tilde{W}_{0h}^0 = W_h^0$, i.e. no missing data at time 0, then the expectation of the product estimator will be somewhat less than that of $\hat{I}_2^{t,0}$ depending on the size of $\tilde{W}_{th}^0$. There is some doubt as to which, if either, of the above expectations is an economically sensible quantity. For the empirical calculations reported in Section 5, we have used the expectation given by (4) as the population index value since it seems to have more intuitive appeal.

Under model (2) $\hat{I}_2^{t,0}$ is also prediction-unbiased while $\hat{I}_1^{t,0}$ is approximately so when stratum sample sizes are large. When (2) fails by omitting regressors, certain sample balance conditions, discussed in Valliant (1987b), are sufficient for model-unbiasedness.

Important differences emerge between the two long-term estimators when variances are considered. Straightforward application of Theorem 2.1 in Royall (1976) shows that the best linear unbiased (BLU) predictor of $I^{t,0}$ is

$$\hat{I}_{BLU}^{t,0} = \hat{I}_2^{t,0} + \Sigma_h \Sigma_{s_{th}} (W_{hi}^0 - \bar{W}_{th}^0) r_{hi}^{t,0} \quad . \qquad (5)$$

where $\bar{W}_{th}^0 = \Sigma_{s_{th}} W_{hi}^0 / n_h$. Under some reasonable conditions the second term on the righthand side of (5) converges in probability to zero under model (2) as $N_h, n_h \to \infty$ and the BLU predictor is asymptotically equivalent to $\hat{I}_2^{t,0}$ (Valliant 1987b). The model variance of $\hat{I}_2^{t,0}$ is

$$\mathrm{var}(\hat{I}_2^{t,0}) = \Sigma_h (W_h^0)^2 \Delta_h^2 / n_h \ .$$

The development for $\hat{I}_1^{t,0}$ is quite involved and is sketched in the appendix. Comparison of the variances of $\hat{I}_1^{t,0}$ and $\hat{I}_2^{t,0}$ is, in general, difficult, but consideration of a special case is informative. Suppose that in each stratum the proportion of units that is common to samples s_{th} and $s_{t'h}$ $(t>t')$ is $(1-\delta)^{t-t'}$, i.e. from one time period to the next the proportionate turnover in sample units is δ. Further, suppose that $\alpha_{th} = \alpha^t$ and that $\rho_h = \rho$. In that special case expression (A.2) in the Appendix can be rewritten in two ways. First, define $X_{\tau k} = [(1-\delta)\rho]^{|\tau-k|}/\alpha^{\tau+k}$, $Y_{\tau k} = [(1-\delta)\rho]^{|\tau-k|}/\alpha^{\tau+k-2}$, $Z_{\tau k} = (1-\delta)^{|\tau-k|}\rho^{|\tau-k+1|}/\alpha^{\tau+k-1}$, and from these $X_t = \Sigma_{\tau=1}^t \Sigma_{k=1}^t X_{\tau k}$, $Y_t = \Sigma_{\tau=2}^t \Sigma_{k=2}^t Y_{\tau k}$ $(t\geq 2)$, $Z_t = \Sigma_{\tau=1}^t \Sigma_{k=2}^t Z_{\tau k}$ $(t\geq 2)$ where $Y_1 \equiv Z_1 \equiv 0$. The two re-expressions of (A.2) are then

$$\mathrm{var}(\hat{I}_1^{t,0}) \doteq \alpha^{2t}\mathrm{var}(\hat{I}_2^{t,0})\{X_t + Y_t - 2Z_t\} \qquad (6)$$

$$= \alpha^2 \mathrm{var}(\hat{I}_1^{t-1,0}) + \alpha^{2t}\mathrm{var}(\hat{I}_2^{t,0})\{X_t^1 + X_t^2 + Y_t^1 - Y_t^2 - 2(Z_t^1 + Z_t^2)\} \ , \qquad (7)$$

where $X_t^1 = \Sigma_{\tau=1}^t X_{\tau t}$, $X_t^2 = \Sigma_{k=1}^{t-1} X_{tk}$, $Y_t^1 = \Sigma_{\tau=2}^t Y_{\tau t}$, $Y_t^2 = \Sigma_{k=2}^{t-1} Y_{tk}$, $Z_t^1 = \Sigma_{\tau=1}^t Z_{\tau t}$, and $Z_t^2 = \Sigma_{k=2}^{t-1} Z_{tk}$. In times of inflation $(\alpha>1)$, expression (6) implies that for $t>1$ the product estimator will be less precise than $\hat{I}_2^{t,0}$ when α^{2t} times the term in braces is greater than or equal to 1. When t=2, for example, α^{2t} times the term in braces in (6) is $\alpha^2[1 + 2\delta\alpha(\alpha-\rho)]$ which is greater than or equal to 1 when $\alpha\geq 1$. From (7) it is apparent that, when $\alpha\geq 1$, the variance of the product estimator will grow as t increases as long as the term in braces in (7) is positive.

The approximate standard error ratio $[\mathrm{var}(\hat{I}_1^{t,0})/\mathrm{var}(\hat{I}_2^{t,0})]^{\frac{1}{2}} = \alpha^t[X_t + Y_t - 2Z_t]^{\frac{1}{2}}$ derived from (6) is plotted versus time in Figure 1 for $\alpha=1.005$, which is equivalent to about 6.2% annual inflation when a period is a month, and for several choices of ρ and δ. For a given level of annual turnover, defined as 1 -

$(1-\delta)^{12}$ for the figure, the standard error ratio increases as ρ decreases. For a given value of ρ, the ratio increases as turnover increases. Generally, the more dissimilar samples are from one time period to another, either because of sample turnover or weak correlation, the more superior $\hat{I}_2^{t,0}$ is to $\hat{I}_1^{t,0}$. As long as δ is nonzero, the product estimator of long-term change is less precise than $\hat{I}_2^{t,0}$ and the gap between the two worsens as t increases.

4.2 Estimators of Short-term Price Change

When stratum sample sizes are large, $\hat{I}_1^{t,m}$ and $\hat{I}_2^{t,m}$ ($m<t$) are both approximately model unbiased under (2) and design unbiased under the pps sampling plan mentioned in Section 4.1.

In discussing precision it is convenient to work with relative variances (relvariances) defined here as $\mathrm{var}(\hat{I}_j^{t,m})/[E(\hat{I}_j^{t,0})/E(\hat{I}_j^{m,0})]^2$ for $j=1,2$. The Appendix gives approximate relvariance formulas under model (2). For the special case in Section 4.1 ($\alpha_{th} = \alpha^t$, $\rho_h = \rho$, $n_{tkh}/n_h = (1-\delta)^{|t-k|}$) the approximate relvariance of $\hat{I}_2^{t,m}$ given by (A.3) reduces to

$$\mathrm{relvar}(\hat{I}_2^{t,m}) = \{\alpha^{-2t} + \alpha^{-2m} - 2\alpha^{-(t+m)} \bullet [(1-\delta)\rho]^{t-m}\}\mathrm{var}(\hat{I}_2^{t,0}). \quad (8)$$

After much algebra (A.4) reduces to

$$\mathrm{relvar}(\hat{I}_1^{t,m}) \doteq \{\Sigma_{\tau=t+1}^{t}\Sigma_{k=m+1}^{t}(X_{\tau k} + Y_{\tau k} - 2Z_{\tau k}\}\mathrm{var}(\hat{I}_2^{t,0}) \quad (9)$$

The ratio of (9) to (8) turns out empirically to depend on t-m rather than the particular values of the two time periods, although we have not shown this to be true for general t and m except in particular cases. For example, if m=t-1 so that 1-period change is being considered, then

$$\mathrm{relvar}(\hat{I}_1^{t,t-1})/\mathrm{relvar}(\hat{I}_2^{t,t-1}) \doteq [1 + 2\rho\delta/(A\alpha)]^{-1} \quad (10)$$

where $A = 1 - 2\rho/\alpha + 1/\alpha^2$. Assuming $\rho>0$, $\alpha>1$, and $\delta>0$, then $A>0$ and (10) will be less than 1, i.e. the product estimator will be more precise than the simpler estimator $\hat{I}_2^{t,t-1}$ for 1-period change. This superiority of the product estimator derives from the numerator and denominator of $\hat{I}_1^{t,t-1}$ being more highly correlated than their counterparts in $\hat{I}_2^{t,t-1}$. For example, in the extreme case $\delta=1$, i.e. complete sample turnover between periods, the numerator and denominator of $\hat{I}_2^{t,t-1}$ are uncorrelated, but the numerator and denominator of the product estimator still contain many elements in common.

Figure 2 is a plot of the relative standard error ratio $[\mathrm{relvar}(\hat{I}_1^{t,m})/\mathrm{relvar}(\hat{I}_2^{t,m})]^{\frac{1}{2}}$ for 1, 3, 6, and 12-period change when $\alpha=1.005$ with a period taken to be a month. For the combinations of correlation and turnover in the figure the product estimator is generally more precise than $\hat{I}_2^{t,m}$. However, for longer time intervals, particularly 6 months or more, the product estimator can be inferior to $\hat{I}_2^{t,m}$ unless the correlation is very high.

5. A SIMULATION STUDY

In order to obtain the approximate results of the preceding sections, we have abstracted a complex situation into one that is much simpler. Thus, it is important to seek empirical verification of the theoretical findings. To do this we conducted a simulation study using data collected by the BLS for the CPI program. A finite population was created consisting of 7668 food items priced monthly during the 3-year period January 1980 through December 1982. The items were part of a national probability sample used for the CPI during that period. The population included seasonal items for which prices were available for only certain months each year, substitute items, and other items which were priced monthly throughout the 3 years. Long-term relatives for individual substitute items were those imputed by BLS for the CPI. An item within an establishment that was phased-in or phased-out during the 3-year period is reflected in the population by having missing prices up to the time of phase-in or by having missing prices after the time of phase-out. The base period, time 0, varied for each item but in all cases was one of the months in the latter half of 1977. A sampling weight was available for each item which was designed to be, under ideal circumstances, proportional to the base period trade value $p_{hi}^0 q_{hi}^0$. These CPI sampling weights were used to compute the base period index weights W_{hi}^0 for the study population.

Table 1 gives various population and sample allocation numbers. Fifteen strata were created based on type of food item. One thousand stratified samples of initial size 200 were selected with the number of sample units allocated to a stratum being roughly proportional to W_h^0. Within each stratum items were sorted by region of the country, type of geographic area (large, medium, or small metropolitan area and nonmetropolitan area). A systematic, random-start sample was then selected in each stratum with probabilities proportional to W_{hi}^0 using the method described in, for example, Hansen, Hurwitz, and Madow (1953 p.343) or Hartley and Rao (1962). Because of seasonal items and items rotated in or out of the original CPI sample, the sample size of useable, i.e. nonmissing, prices varied over the 36 months. The average over the 1000 stratified samples and the 36 months of the sample size of useable prices was 136 with the average for individual time periods ranging from 103 to 173. In the pps sampling plan used here, the expected percentage of overlap in useable items for different months depends on stratum and the distance between months. For example, in stratum 3 (beef) the expected overlaps between month 1 and months 12, 24, and 36 are 47, 44, and 20%. For stratum 13 (processed fruit) the same expected overlaps are 53, 53, and 16%.

The population and the samples depart in a number of ways from the assumptions used to derive the earlier theory. As noted before, the sample size of useable items varies among months rather than being constant. The proportion of turnover in the sample from one month to the next is not a constant δ. A first-order autocorrelation model does not appear to be complex enough to describe the population well. Across all strata, for example, the simple correlations of the long-term relative for month 1 with those for months 2, 12, 24, and 36 are .89, .80, .68, and .56, indicating that the correlation does not die off as rapidly as would be expected under a first-order model. Because of these departures from the theoretical assumptions, the simulation study provides a reasonably stringent test of the robustness of the theoretical findings.

From each of the 1000 samples the long-term estimators $\hat{I}_1^{t,0}$ and $\hat{I}_2^{t,0}$ $(t=1,2,\ldots,36)$ and the short-term estimators of 1, 3, 6, and 12-month change, $\hat{I}_1^{t,m}$ and $\hat{I}_2^{t,m}$ $(m=t-1,t-3,t-6,t-12\ ;\ t-m\geq 1)$ were computed. Across the samples we then computed relative biases, defined as $\Sigma(\hat{I}-I)/I$ where the summation is over the 1000 samples, $\hat{I}$ is one of the long or short-term estimators, and I is the corresponding population index accounting for missing data, defined by (4), for long-term change or ratios of indexes defined by (4) for short-term change. Similarly, square roots of relative mean squared errors (rmse's) were computed as $[\Sigma(\hat{I}-I)^2/I^2]^{\frac{1}{2}}$.

The results are plotted versus time in Figure 3 for the full population estimates and in Figures 4 and 5 for two individual strata — beef and fresh fruit. Each figure contains results for estimates of long-term change and 1,6,and 12-month short-term change. Graphs for 3-month change were similar to those for 6-month and are omitted. The simple estimator $\hat{I}_2^{t,m}$ (m=0,1,6,12) is nearly unbiased for all values of t. The product estimator of long-term change is often negatively biased in Figure 3 as predicted in Section 4.1. For short-term change the bias of the product estimator is much more erratic than that of the simple estimator in each of Figures 3-5.

For long-term change $\hat{I}_2^{t,0}$ is consistently precise while the root rmse of $\hat{I}_1^{t,0}$ increases with t as would be true under model (2). For 1 and 6-month change the product estimator is generally more precise than the simple estimator for full population estimates and for the individual beef and fresh fruit strata. For 12-month change the relationship is less consistent. $\hat{I}_2^{t,0}$ is more precise more often than $\hat{I}_1^{t,0}$ for the full population and for fresh fruit, but the reverse is true for beef.

Although precision was discussed here in mean squared errors, variances of estimators over the 1000 samples were also examined and showed the same general patterns noted above.

6. CONCLUSION

Two estimators of fixed base Laspeyres price indexes are the product estimator, which incorporates sample price change data from all periods from the base to the current, and a simpler estimator which uses data only from the current period's sample. Both estimators have strengths and weaknesses. For long-term change the simpler estimator is preferable since it has a smaller relative variance while the product estimator has a relative variance that increases with time. For short-term change of less than 1 year the product estimator appears to be more precise. At some point the length of time over which change is measured will become long enough that the superiority of the product estimator for short-term change will disappear. Based on the theoretical and empirical work here that point may be about 1 year.

The contrasts between the properties of the product and the simple estimators suggest that further research can be usefully done on compromise estimators designed to perform reasonably well for both long and short-term change. Statisticians at the BLS are currently investigating estimators, which are similar to the product estimator in using sample data from a number of time periods, but are intended to improve the estimates of long-term change with minimal deterioration of the estimates of short-term change.

APPENDIX : LARGE SAMPLE VARIANCE APPROXIMATIONS FOR INDEX ESTIMATORS

A.1 Approximate Variance of the Product Estimator of Long-term Change

The usual first-order approximation gives

$$\Pi_{\tau=1}^{t}\hat{R}_h^{\tau,\tau-1} \doteq \Pi_{\tau=1}^{t}(\alpha_{\tau h}/\alpha_{\tau-1,h}) + \Sigma_{\tau=1}^{t}(\alpha_{th}/\alpha_{\tau h})(\bar{r}_{\tau h}^{\tau} - \alpha_{\tau h}) - \Sigma_{\tau=1}^{t}(\alpha_{th}/\alpha_{\tau-1,h})(\bar{r}_{\tau h}^{\tau-1} - \alpha_{\tau-1,h}). \quad (A.1)$$

The approximate variance involves the variances and covariances of different sample averages of long-term relatives. We sketch the calculation of $\mathrm{cov}(\bar{r}_{\tau h}^{\tau},\bar{r}_{kh}^{k-1})$, $\tau\neq k$. Define $s_{\tau kh}$ to be the set of sample units in stratum h that are common to periods τ and k, $\tilde{s}_{\tau h}$ to be the complement of $s_{\tau kh}$ within $s_{\tau h}$, and $\tilde{s}_{kh}$ to be the complement of $s_{\tau kh}$ within s_{kh}. We have, after some algebra,

$$\begin{aligned}\mathrm{cov}(\bar{r}^{\tau}_{\tau h}, \bar{r}^{k-1}_{kh}) &= \mathrm{cov}[\Sigma_{s_{\tau kh}} r^{\tau}_{hi} + \Sigma_{\tilde{s}_{\tau h}} r^{\tau}_{hi}\,, \\ &\qquad \Sigma_{s_{\tau kh}} r^{k-1}_{hi} + \Sigma_{\tilde{s}_{kh}} r^{k-1}_{hi}]/n^2_h \\ &= n_{\tau kh}\rho_h^{|\tau-k+1|}\Delta^2_h/(n^2_h\alpha_{\tau h}\alpha_{k-1,h})\end{aligned}$$

when k>1 and 0 if k=1, where $n_{\tau kh}$ is the number of items in $s_{\tau kh}$. Other terms are obtained by similar straightforward, though laborious calculations. The approximate variance is

$$\mathrm{var}(\hat{I}^{t,0}) \doteq \Sigma_h(W^0_h\Delta_h\alpha_{th})^2 V_{tth}\,, \tag{A.2}$$

where

$$V_{tth} = \left\{\Sigma^t_{\tau=1}\Sigma^t_{k=1}\lambda_{\tau kh}\frac{\rho_h^{|t-k|}}{\alpha_{\tau h}\alpha_{kh}} + \Sigma^t_{\tau=2}\Sigma^t_{k=2}\lambda_{\tau kh}\frac{\rho_h^{|t-k|}}{\alpha_{\tau-1,h}\alpha_{k-1,h}} - \Sigma^t_{\tau=1}\Sigma^t_{k=2}\lambda_{\tau kh}\frac{\rho_h^{|\tau-k+1|}}{\alpha_{\tau h}\alpha_{k-1,h}} - \Sigma^t_{\tau=2}\Sigma^t_{k=1}\lambda_{\tau kh}\frac{\rho_h^{|\tau-k-1|}}{\alpha_{\tau-1,h}\alpha_{kh}}\right\}$$

with $\lambda_{\tau kh} = n_{\tau kh}/n^2_h$. The subscripts tt identify the upper limits of the double summations in the expression for V_{tth}.

A.2 Approximate Relative Variances of Estimators of Short-term Change

The standard delta method relvariance approximation for $\hat{I}^{t,m}_2$ (t>m) is

$$\mathrm{relvar}(\hat{I}^{t,m}_2) \doteq \Sigma_h(W^0_h\Delta_h)^2 n^{-1}_h\{(\Sigma_h W^0_h\alpha_{th})^{-2} + (\Sigma_h W^0_h\alpha_{mh})^{-2} - 2\lambda_{tmh}\rho_h^{t-m}[(\Sigma_h W^0_h\alpha_{th})(\Sigma_h W^0_h\alpha_{mh})]^{-1}\}. \tag{A.3}$$

The computations for the approximate relvariance of $\hat{I}^{t,m}_1$ are lengthy but routine. Approximating $\Pi^t_{\tau=1}\hat{R}^{\tau,\tau-1}_h$ and $\Pi^m_{\tau=1}\hat{R}^{\tau,\tau-1}_h$ as in (A.1) and working out the necessary covariances leads to

$$\mathrm{relvar}(\hat{I}^{t,m}_1) \doteq \mathrm{relvar}(\hat{I}^{t,0}_1) + \mathrm{relvar}(\hat{I}^{m,0}_1) - 2[(\Sigma_h W^0_h\alpha_{th})(\Sigma_h W^0_h\alpha_{mh})]^{-1}V_{tmh}\,. \tag{A.4}$$

The term V_{tmh} is defined analogously to V_{tth} given in Section A.1.

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Table 1. Universe and sample numbers for the population of food items.

Stratum	N_h	N_h/N (%)	W_h^0 (%)	n_h	n_h/N_h (%)
1 Cereal	285	3.7	2.9	8	2.8
2 Bakery goods	762	9.9	6.9	14	1.8
3 Beef	962	12.5	16.1	24	2.5
4 Pork	705	9.2	7.3	14	2.0
5 Other meat	354	4.6	4.4	10	2.8
6 Poultry	299	3.9	6.0	12	4.0
7 Fish	201	2.6	3.4	8	4.0
8 Eggs	173	2.3	4.4	10	5.8
9 Milk	331	4.3	10.1	18	5.4
10 Other dairy	476	6.2	5.7	12	2.5
11 Fresh fruit	1078	14.1	11.3	22	2.0
12 Fresh vegetables	979	12.8	10.4	20	2.0
13 Processed fruit	266	3.5	2.5	8	3.0
14 Processed veg.	369	4.8	4.0	10	2.7
15 Sugar	428	5.6	4.6	10	2.3
Total	7668	100.0	100.0	200	2.6

Note : The composition of the study population is not intended to to duplicate that of the U.S. CPI population. Values of W_h^0 may differ substantially from those in the actual CPI.

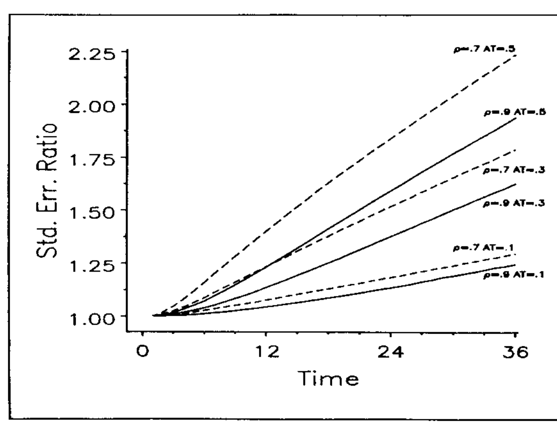

Figure 1. Theoretical standard error ratios of the product estimator to the simple estimator for long-term price change plotted versus time. The annual inflation rate used is 6.2% (α=1.005). AT is the annual turnover rate.

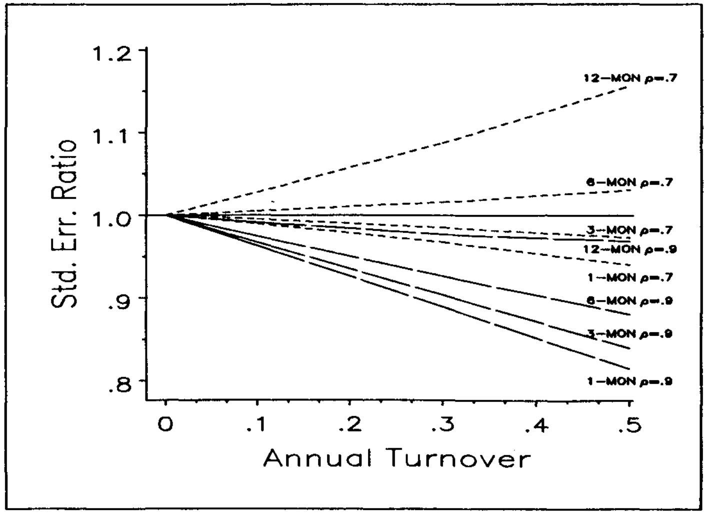

Figure 2. Theoretical standard error ratios of the product estimator to the simple estimator for 1,3,6, and 12-month short-term price change plotted versus annual turnover rate. The annual inflation rate used is 6.2% (α=1.005).

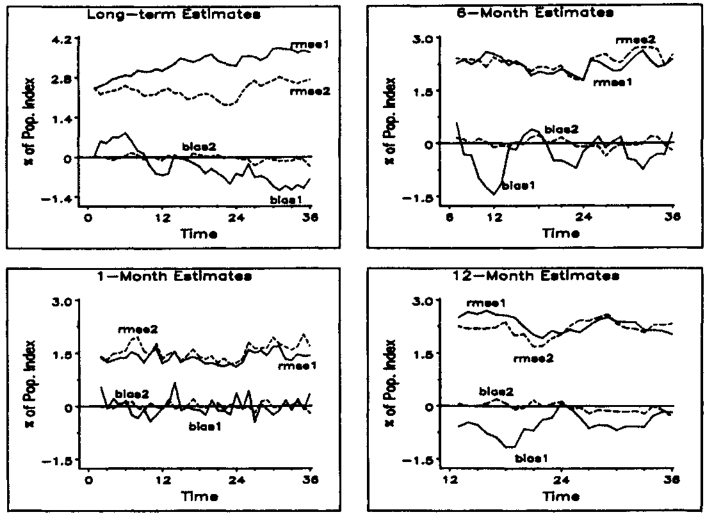

Figure 3. Relative biases and square roots of relative mean squared errors of the product estimator and the simple estimator in 1000 stratified samples. Results are for estimates for the full population.

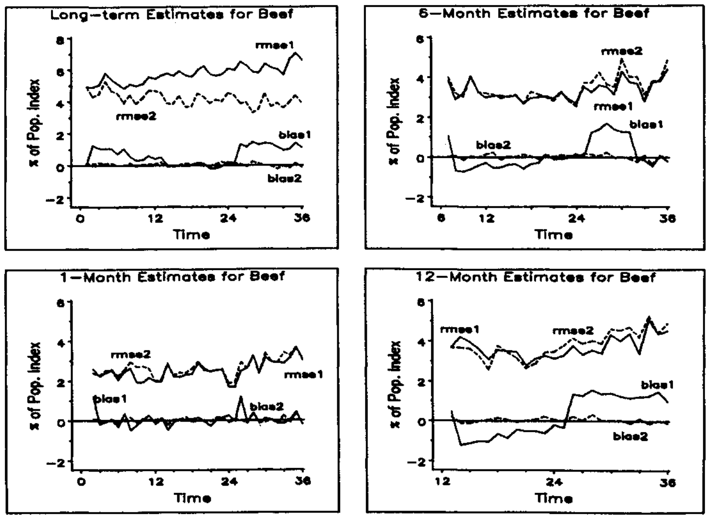

Figure 4. Relative biases and square roots of relative mean squared errors of the product estimator and the simple estimator in 1000 stratified samples. Results are for estimates for the stratum of beef items.

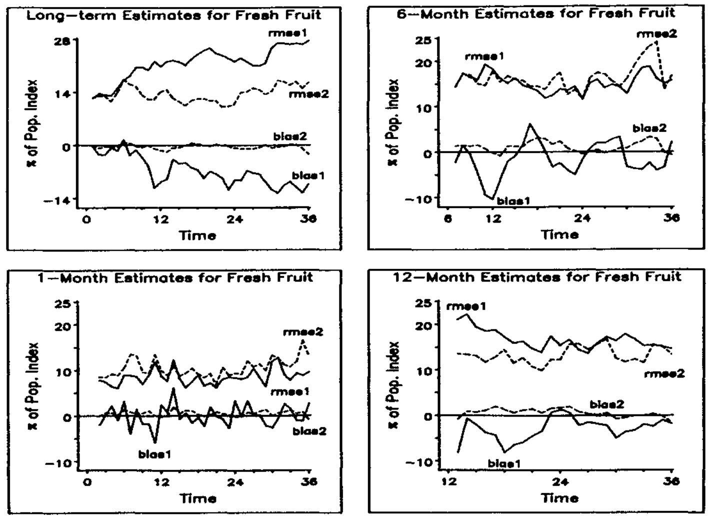

Figure 5. Relative biases and square roots of relative mean squared errors of the product estimator and the simple estimator in 1000 stratified samples. Results are for estimates for the stratum of fresh fruit items.