STATISTICAL PROPERTIES OF LINEAR SURVEY ESTIMATORS WITH INTEGER VALUED RANDOM VARIABLE WEIGHTS

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INTRODUCTION

Large statistical surveys often require linear estimators with integer weights. Integer weights are required because of the survey data processing system used to tabulate the data. Integer weight linear estimators reduce cost by simplifying numerically and operationally the computation of survey estimates. Rarely, however, will an estimator in a complex survey have weights which are naturally integers. To achieve a gain in data processing efficiency, noninteger constant weights must be converted to integer weights. Converting the weights reduces survey data processing costs; but, it increases the total survey error of the estimator by adding a nonsampling variance or a nonsampling bias. This conversion is done by making the weight an integer valued random variable. The new estimator is a linear estimator with integer random variable weights. This is a special case of the general problem of having a linear estimator with random variable weights. An alternative to integer weighting is to reduce sample size and use the money saved to develop a data processing system which can handle real number weights.

A major argument given for using integer weights is that they simplify tabulation by eliminating the need to round frequency tables for count data. For example, Table 1 gives a frequency distribution of a sample characteristic --hired farm labor--from the 1978 Census of Agriculture.

Table 1. Number of Farms Reporting Hired Farm Workers in Charles County, Maryland, in 1978 Census of Agriculture

Characteristic	Number
Farms	369
Farms with	
1 worker	55
2 workers	57
3 or 4 workers	108
5 to 9 workers	93
10 workers or more	56

If noninteger weights were used to produce Table 1, the total number of farms, and the number of farms in each frequency class would be rounded to an integer. The rounded total number of farms generally would not be the same as the sum of the rounded counts for each individual frequency class. Integer weights would, on the other hand, always give a total count of the number of farms which agrees with the sum of the counts from a set of mutually exclusive, exhaustive classes of farms.

A second argument given for using integer weights is that integer weights along with integer data values allow the use of integer arithmetic in computer programs. This increases computational efficiency and reduces the computer cost.

A special case of the integer weight linear estimator problem has been studied in connection with the 1970 and 1980 Census of Population and Housing. Hanson (1969) and Thompson (1978) considered a restricted situation and studied the problem. No general solution to the problem was found.

BACKGROUND

Given a population of elementary units, $\underline{U}$, with values of a characteristic, $\underline{X}$, then a population parameter, θ, is a function of the population characteristics:

$$\theta = f(\underline{X})$$

$$\underline{U}^T = (U_1, U_2, \ldots, U_i, \ldots, U_N)$$

$$\underline{X}^T = (X_1, X_2, \ldots, X_i, \ldots, X_N)$$

A sample of n population units, $\underline{u}$, is selected from the population $\underline{U}$ using a probability sampling plan. The data collected on sample units, $\underline{u}$, are given by the observed value vector, $\underline{x}$. The general form of the linear estimator of the parameter, θ, which has weights which are constant values is given by (1). The expected value and variance of the estimator (1) is given in (2) and (3), respectively. The "best" linear unbiased estimator, BLUE, of the parameter, $\tilde{\theta}$, is an equation of the form of (1).

$$\tilde{\theta} = \underline{W}^T \underline{x} \quad (1)$$

$$E(\tilde{\theta}) = \underline{W}^T \underline{\mu}_x \quad (2)$$

$$\sigma^2_{\tilde{\theta}} = \underline{W}^T \Sigma_x \underline{W} \quad (3)$$

where

$\tilde{\theta}$ = The estimator of the parameter θ.

$\underline{W}$ = An n by 1 vector of constant weights of the vector of sample observations $\underline{x}$.

$$\underline{W}^T = (W_1, W_2, \ldots, W_i, \ldots, W_n)$$

$\underline{x}$ = An n by 1 vector of sample observations of the characteristic x on the vector of sample population units, $\underline{u}$.

$$\underline{x}^T = (x_1, x_2, \ldots, x_i, \ldots, x_n)$$

$\underline{\mu}_x$ = An n by 1 vector of expected values of the sample vector $\underline{x}$.

$$\underline{\mu}^T_x = (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_i}, \ldots, \mu_{x_n})$$

Σ_x = An n by n variance-covariance matrix of the sample vector $\underline{x}$. The variance-covariance matrix is determined by the survey population and the sample design.

$$\Sigma_x = E(\underline{x}-\underline{\mu}_x)(\underline{x}-\underline{\mu}_x)^T$$

There are an infinite number of possible linear estimators for the parameter θ. In most surveys there will be a preferred estimator, often the BLUE, for each parameter of a characteristic. In many surveys the same weights will be used for all linear estimators for all characteristics of a parameter.

RANDOM WEIGHT ESTIMATOR

If the weights associated with the estimator (1) are random variables instead of constants, a linear estimator with different properties is produced. Random variable weights change the statistical properties of an estimator of θ. The new estimator with random variable weights is given in (4). A special case is the situation of a random variable with integer values.

The statistical characteristics of the random weight estimator (4) depend, like the constant weight estimator (1), on the sampling plan and survey population. In addition the statistical characteristics of the estimator (4) depend on the weighting scheme used to produce weights. When weights are created using a probability weighting scheme statistically independent of the sample values, $\underline{x}$, the expected value and variance of the estimator (4) are given in (5) and (6), respectively. The bias of the random weight estimator as an estimator of the expected value of the constant weight estimator is given in (8).

$$\hat{\theta} = \underline{w}^T \underline{x} \quad (4)$$

$$E(\hat{\theta}) = \underline{\mu}_w^T \underline{\mu}_x \quad (5)$$

$$\sigma^2_{\hat{\theta}} = \underline{\mu}_w^T \Sigma_x \underline{\mu}_w + tr(\Sigma_x \Sigma_w) + \underline{\mu}_x^T \Sigma_w \underline{\mu}_x \quad (6)$$

$$\sigma^2_{\hat{\theta}} = \sigma^2_{\tilde{\theta}} + (\underline{\mu}_w - \underline{W})^T \Sigma_x (\underline{\mu}_w + \underline{W})$$

$$+ \; tr(\Sigma_x \Sigma_w) + \underline{\mu}_x^T \Sigma_w \underline{\mu}_x$$

when $\underline{\mu}_w = \underline{W}$

$$\sigma^2_{\hat{\theta}} = \sigma^2_{\tilde{\theta}} + tr(\Sigma_x \Sigma_w) + \underline{\mu}_x^T \Sigma_w \underline{\mu}_x \quad (7)$$

where

$\hat{\theta}$ = The random weight linear estimator of the parameter θ.

$\underline{w}$ = The vector of random variable weights of the sample units $\underline{u}$.

$$\underline{w}^T = (w_1, w_2, \ldots, w_i, \ldots, w_n)$$

$\underline{\mu}_w$ = The expected value of the weight vector $\underline{w}$.

$$\underline{\mu}_w^T = (\mu_{w_1}, \mu_{w_2}, \ldots \mu_{w_i}, \ldots, \mu_{w_n})$$

$\sigma^2_{\tilde{\theta}}$ = The variance of the constant weight estimator when the random weight vector is an unbiased estimator of $\underline{W}$.

Σ_w = The n by n variance-covariance matrix of the random weight vector, $\underline{w}$.

$$\Sigma_w = E(\underline{w} - \underline{\mu}_w)(\underline{w} - \underline{\mu}_w)^T$$

The expected value and variance of the random weight estimator are developed in appendices 1 and 2. The statistical properties are the same for integer and noninteger value random variables.

Since the random weight linear estimator is used as a replacement for the constant weight estimator, the bias of the random weight estimator compared to the expected value of the constant weight estimator is given in (8).

$$Bias_{E\tilde{\theta}}(\hat{\theta}) = E(\hat{\theta}) - E(\tilde{\theta}) \quad (8)$$

$$= (\underline{\mu}_w - \underline{W})^T \underline{\mu}_x$$

The random weight estimator and the constant weight estimator have the same expected value when the random weight vector is an unbiased estimator of the constant weight vector.

ESTIMATION OF THE SAMPLING VARIANCE

The true variance of the random weight estimator (4) is given in (6). In estimating the variance (6), the expected value and the variance-covariance matrix of $\underline{w}$ are known.

When $\underline{w}$ is an unbiased estimator of $\underline{W}$, an unbiased estimator of the random weight variance estimator (6) is given in (9).

$$\hat{\sigma}^2_{\hat{\theta}} = \hat{\sigma}^2_{\tilde{\theta}} + \underline{x}^T \Sigma_w \underline{x} \quad (9)$$

where:

$\hat{\sigma}^2_{\hat{\theta}}$ = An unbiased estimate of $\sigma^2_{\hat{\theta}}$

$\underline{\mu}_w$ = The n by 1 vector of expected values of random weight vector, $\underline{w}$.

Σ_w = The n by n variance-covariance matrix of random weight vector, $\underline{w}$.

$\underline{x}$ = The n by 1 vector of sample values.

$\hat{\sigma}^2_{\tilde{\theta}}$ = An unbiased estimator of the variance of the constant weight estimator.

The proof is given in Appendix 3.

SIMPLE RANDOM SAMPLING WITHOUT REPLACEMENT

The general random weight estimator (4) can be used with any sampling plan, any linear estimator, and any weighting scheme. However, when either the sampling plan or the weighting scheme have regular statistical properties the form of the estimator can be simplified. Regular statistical properties refers to statistical characteristics such as either all weights or all variables having the same expected value and variance or the covariance being the same for all pairs of weights or all pairs of variable values. The simplified form of the estimator for a basic sampling plan or weighting scheme can be used to illustrate the general properties of the random weighting estimator.

When a probability sample is selected using a simple random sample without replacement sampling plan, the constant weight estimator and the random weight estimator of the population total are given in (10) and (11), respectively. The constant weight estimator is the best linear unbiased estimator of the population total. The expected value of the two estimators (10) and (11) is the population total whenever the random weight vector, $\underline{w}$, is an unbiased estimator of the constant weight vector, $\underline{W}$.

Simple random sampling without replacement produces a variance-covariance matrix of the sample vector that is a pattern matrix of the form given in (12). The regular form of the variance-covariance matrix changes the expression of the variance of the two estimators (1) and (4). The variance of the constant weight estimator is given in (13).

When the weighting scheme selected for constructing weighting is regular--all weights have the same expected value and variance, and the covariance between any two weights is the same--the variance of the random weight estimator can be simplified. When the random weight $\underline{w}$ is an unbiased estimator of $\underline{W}$, the variance of the random weight estimator is given in (14). The expression of the variance of the random weight estimator (14) shows the effect of random weighting on the total variance of the estimate.

Relative efficiency can be used to compare the constant weight estimator (10) and the random weight estimator (11) of a population total. The relative efficiency of the two estimators (15) is expressed in terms of the population size, the sample size, the relative variance of the population, the relative variance of the random weight, and the correlation between random variable weights.

$$x'_{srs,\ constant} = W_{srs}\underline{j}^T\underline{x} \qquad (10)$$

$$x'_{srs,\ random} = \underline{w}^T\underline{x} \qquad (11)$$

$$\Sigma_{x,\ srs} = \left[(1 + 1/n)\ I - (1/n)\ J\right]\sigma_x^2 \qquad (12)$$

$$\sigma^2_{x'_{srs,\ constant}} = n\mu_w^2(N-n)N^{-1}\sigma_x^2 \qquad (13)$$

when $\underline{\mu}_w = W_{srs}\underline{j}$ and $\Sigma_w = \left[(1-\rho_w)I+\rho_w J\right]\sigma_w^2$

$$\sigma^2_{x'_{srs,\ random}} = \sigma^2_{x'_{srs,constant}} + n\sigma_w^2(\mu_x^2+\sigma_x^2) \qquad (14)$$

$$+\ n(n-1)\ \rho_w\sigma_w^2(\mu_x^2 - \sigma_x^2 N^{-1})$$

Relative Efficiency $(x'_{srs,\ random} : x'_{srs,\ constant})$ (15)

$$= \sigma^2_{x'_{srs,\ constant}} / \sigma^2_{x'_{srs,\ random}}$$

$$= \left[1+V_w^2 N(N-n)^{-1}V_x^{-2}+(n-1)\rho_w V_w^2(V_x^{-2}-N^{-1})\right]^{-1}$$

Where

$x'_{srs,\ constant}$ = An unbiased estimate of population total under simple random sampling with constant weights.

$x'_{srs,\ random}$ = An unbiased estimate of population total under simple random sampling with random weights.

W_{srs} = The constant weight expansion factor for simple random sampling.

$\underline{j}$ = An n by 1 vector of 1's.

$\underline{j}^T = (1,1,\ldots,1)$

$\Sigma_{x,\ srs}$ = The n by n variance-covariance matrix of the sample vector $\underline{x}$ when $\underline{x}$ is selected using a simple random sampling without replacement sampling plan.

I = An n by n identity matrix.

J = An n by n matrix of 1's.

ρ_w = Correlation between random weights when the correlation between all pairs of weights is the same.

V_x^2 = The relative variance of the population characteristic X.

$V_x^2 = \sigma_x^2 / \mu_x^2$

V_w^2 = The relative variance of the random weight.

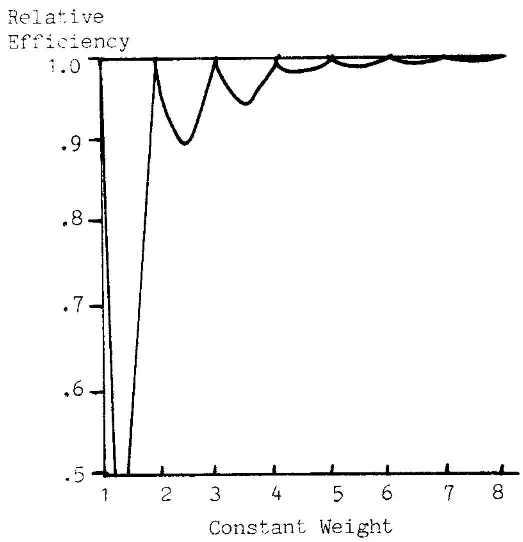
$V_w^2 = \sigma_w^2 / \mu_w^2$

Table 2 shows the relative efficiency of the integer valued random weight estimator and the constant weight estimator for simple random sampling without replacement. The integer valued random weight estimator uses the weighting scheme described in equation (16). The correlation between weights assigned to different sampling units is zero. Figure 1 shows graphically the relative efficiency of the two estimators over a range of values. The variability of the population surveyed, V_x^2, has a large effect on the relative efficiency of the two estimators.

Table 2. Relative efficiency of integer valued random weight and constant weight estimator of a population total for simple random sampling without replacement for different weights and different populations

Weight	Population Relative Variance--V_x^2			
	.5	1.0	2.0	4.0
1.5	.6000	.7500	.8571	.9231
2.5	.8823	.9375	.9677	.9836
3.5	.9459	.9722	.9677	.9929
4.5	.9692	.9844	.9924	.9960
5.5	.9802	.9900	.9950	.9975
10.5	.9950	.9975	.9987	.9994
15.5	.9978	.9989	.9994	.9997
20.5	.9988	.9993	.9997	.9998

Figure 1. Relative efficiency of interior weight and constant weight estimator for sample random sampling without replacement.



The random variable weight used is the one described in equation (16) in the example below.

EXAMPLE

Integer weighting is frequently used in major statistical sample surveys and censuses. Integer weighting was used in the estimator for the 1978 Census of Agriculture Area Sample.

Sample Design--The sample design for the 1978 Census of Agriculture Area Sample was a stratified sample with a single systematic sample selected within each stratum. A sample unit is an area segment. Each segment was canvassed and all farms in the segment enumerated. The estimator of the variance assumes the population is randomly ordered. The sample is considered a simple random sample without replacement. The sample within each stratum was selected with a constant rate for each state. The sampling interval or expansion factor was generally not an integer. For the state of Delaware, the population size, number of samples selected, and expansion factors (sampling interval) for each stratum are given in Table 3.

Table 3. Stratum Characteristics for Delaware in the 1978 Census of Agriculture Area Sample

Stratum h	Number of Segments: Population N_h	Number of Segments: Sample n_h	Expansion Factor W_h
1	207	45	4.6
2	83	20	4.13
3	92	9	10.13
4	42	2	21.00
5	231	3	76.79

Weighting--Integer weights were assigned independently to each segment. For each sample unit--an area segment--a continuous uniform (0,1) random variable was selected. If the random number selected was less than or equal to the noninteger part of the segment weight, the weight was rounded up. If the random number was greater than the integer part of the random weight, the weight was rounded down.

The integer random variable weight was of the form given in (16). The expected value and the variance of the random variable is given in (17) and (18). The variable ε is a Bernouli, zero or one, random variable with an expected value equal to the noninteger part of the weight.

$$W = [W] + P$$

$$w = [W] + \varepsilon \tag{16}$$

$$Ew = W + P \tag{17}$$

$$\sigma_w^2 = P(1-P) \tag{18}$$

where

$[W]$ = Largest integer smaller than W.

$\varepsilon \sim (P, \sigma_w^2)$

ε = 1 or 0.

For example, when a weight is 4.13, the remainder is .13. If the selected random number associated with the segment is .72, the weight is rounded down to the integer 4. If the selected random number is .1, the weight is rounded up to the integer 5. The expected value of a weight is 4.13.

Estimators--The constant weight estimator of a population total in the 1978 Census of Agriculture Area Sample is given by equation (19).

The integer weight estimator for Delaware is given in (20). The integer weight estimator (20) and the constant weight estimator (19) were compared for Delaware for four major characteristics--number of farms, land in farms, cropland harvested, and sales. The relative efficiency of the two estimators was calculated. The difference in the estimate and a comparison of their sampling variances are given in Tables 4 and 5.

$$x''_{constant} = \sum_{h=1}^{5} \sum_{s=1}^{n_h} W_{hs} \sum_{f=1}^{W_{hs}} X_{hsf} \tag{19}$$

$$x''_{integer} = \sum_{h=1}^{5} \sum_{s=1}^{n_h} w_{hs} \sum_{f=1}^{w_{hs}} X_{hsf} \tag{20}$$

where

$x''_{constant}$ = Estimate of the total value of characteristic x in Delaware using constant weight.

$x''_{integer}$ = Estimate of the total value of a characteristic x in Delaware using integer valued random weights.

h = Stratum.

s = Segment.

f = Farm.

n = Number of segments in stratum h.

n_h = Number of farms in segment s of stratum h.

W_{hs} = Weight associated with segment s in stratum h.

w_{hs} = Integer weight associated with segment s and stratum h.

$$Ew_{hs} = W_{hs}$$

The estimator (19) is the best linear unbiased estimator of the population total of census farms not included on the mail list in Delaware. The data processing system used in the 1978 Census of Agriculture Area Sample required that each expansion factor be an integer. Since the sampling interval used to select the sample was not an integer, the true expansion factors for each farm enumerated were not integers. Each noninteger weight was converted to an integer. The estimator (20) is the linear estimator using these integer weights to estimate the population total.

For four selected characteristics--number of farms, land in farms, cropland harvested, and sales--in the state of Delaware, estimators for the two estimators are close. The relative efficiency of the integer weight and constant weight estimator varies by between .9698 and .9877. Integer weighting increased the variance by 1.2 to 3.0 percent.

Table 4. Comparison of Estimated Delaware Totals for Selected Characteristics in 1978 Census of Agriculture Area Sample for Constant Weight and Integer Weight Estimates

Charac-teristic	Estimates of Totals		Difference	
	Constant Weight Estimator x'	Integer Weight Estimator x"	Absolute d=x" - x'	Relative 100d ÷ x'
All Farms	232.9	234	+1.1	+0.47
Land in Farms (Acres)	9314.8	9,399	+84.2	+0.90
Cropland Harvested (Acres)	3718.8	3,661	-57.8	-1.55
Sales (Dollars)	4,660,864.3	4,755,601	+94,763.7	+2.03

Table 5. Comparison of Variances of Constant Weight and Integer Weight Estimators for 1978 Census of Agriculture Area Sample in Delaware

Charac-teristic	Absolute Variance		Relative Efficiency Integer Weight To Constant Weight Estimator
	Constant Weight Estimator	Integer Weight Estimator	
All Farms	1,351	1,393	.9698
Land in Farms (Acres)	8,377,831	8,481,810	.9877
Cropland Harvested (Acres)	1,113,132	1,146,533	.9709
Sales ($1,000)	$1,915,939\times10^6$	$1,956,281\times10^6$	.9794

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APPENDICES

Appendix 1 gives a proof of the covariance between the product of two variables. Appendix 2 gives a proof of the variance of the random weight linear estimator. Appendix 3 proves that if there is an unbiased estimator of the variance of the constant weight estimator then there is an unbiased estimator of the variance of the random weight estimator.

Appendix 1

Let $\underline{x}$ and $\underline{y}$ be statistically independent random vectors with the following expected values and variance-covariance matrix.

$$\underline{x} = (x_1, x_2, \ldots, x_i, \ldots, x_n)$$

$$\underline{y} = (y_1, y_2, \ldots, y_i, \ldots, y_n)$$

$$E\underline{x}^T = \underline{\mu}_x^T = (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_i}, \ldots, \mu_{x_n})$$

$$E\underline{y}^T = \underline{\mu}_y^T = (\mu_{y_1}, \mu_{y_2}, \ldots \mu_{y_i}, \ldots, \mu_{y_n})$$

$$\Sigma_x = E(\underline{x}-\underline{\mu}_x)(\underline{x}-\underline{\mu}_x)^T = ((\sigma_{x_i x_j}))$$

$$\Sigma_y = E(\underline{y}-\underline{\mu}_y)(\underline{y}-\underline{\mu}_y)^T = ((\sigma_{y_i y_j}))$$

The random variable $z_i = x_i y_i$ has the following properties:

$$Ez_i = Ex_i y_i = Ex_i Ey_i = \mu_{x_i}\mu_{y_i} = \mu_{z_i}$$

$$\sigma_{z_i z_j} = E(z_i - Ez_i)(z_j - Ez_j)$$

$$= E(x_i y_i - Ex_i y_i)(x_j y_j - Ex_j y_j)$$

$$= E(x_i y_i x_j y_j) - Ex_i y_i Ex_j y_j$$

$$= Ex_i x_j Ey_i y_j - Ex_i Ex_j Ey_i Ey_j$$

$$= (\sigma_{x_i x_j} + Ex_i Ex_j)(\sigma_{y_i y_j} + Ey_i Ey_j) - Ex_i Ex_j Ey_i Ey_j$$

$$= Ex_i Ex_j \sigma_{y_i y_j} + Ey_i Ey_j \sigma_{x_i x_j} + \sigma_{x_i x_j}\sigma_{y_i y_j}$$

$$= \mu_{x_i}\mu_{x_j}\sigma_{y_i y_j} + \mu_{y_i}\mu_{y_j}\sigma_{x_i x_j} + \sigma_{x_i x_j}\sigma_{y_i y_j}$$

when $i = j$

$$\sigma^2_{x_i y_i} = \mu^2_{x_i}\sigma^2_{y_i} + \mu^2_{y_i}\sigma^2_{x_i} + \sigma^2_{x_i}\sigma^2_{y_i} \qquad (21)$$

Equation (21) is the variance of the product of two independent random variables derived by L.A. Goodman (1960).

Appendix 2

$$\hat{\theta} = \underline{w}^T\underline{x} = \sum_{i=1}^{n} w_i x_i$$

$$\sigma^2_{\hat{\theta}} = \sum_{i=1}^{n}\sigma^2_{w_i x_i} + \sum_{i \neq j}^{n}\sum^{n}\sigma_{w_i x_i, w_j x_j}$$

By Appendix 1 since w_i and x_i are independent

$$\sigma^2_{\hat{\theta}} = \sum_{i=1}^{n}(\mu^2_{x_i}\sigma^2_{w_i} + \mu^2_{w_i}\sigma^2_{x_i} + \sigma^2_{x_i}\sigma^2_{w_i})$$

$$+ \sum_{i \neq j}^{n}\sum^{n}(\mu_{x_i}\mu_{x_j}\sigma_{w_i w_j} + \mu_{w_i}\mu_{w_j}\sigma_{x_i x_j} + \sigma_{w_i w_j}\sigma_{x_i x_j})$$

Rearranging the order

$$\sigma^2_{\hat{\theta}} = (\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{w_i}\mu_{w_j}\sigma_{x_i x_j})$$

$$+ (\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{x_i x_j}\sigma_{w_i w_j})$$

$$+ (\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{x_i}\mu_{x_j}\sigma_{w_i w_j})$$

$$\sigma^2_{\hat{\theta}} = \underline{\mu}_w^T \Sigma_x \underline{\mu}_w + tr(\Sigma_x \Sigma_w) + \underline{\mu}_x^T \Sigma_w \underline{\mu}_x$$

Appendix 3

If $\underline{x}$ and $\underline{w}$ are statistically independent random vectors

$\underline{x} \sim (\underline{\mu}_x, \Sigma_x)$ $\qquad$ $\underline{w} \sim (\underline{\mu}_w, \Sigma_w)$

Then by Theorem 1, Sector 25, Searle (1971),

$$E\underline{x}^T \Sigma_w \underline{x} = tr(\Sigma_w \Sigma_w) + \underline{\mu}_x^T \Sigma_w \underline{\mu}_x$$

When $\hat{\sigma}^2_{\tilde{\theta}}$ is an unbiased estimator of the variance of $\tilde{\theta}$, then $\hat{\sigma}^2_{\hat{\theta}}$ will be an unbiased estimator of the variance of $\hat{\theta}$.

$$\hat{\sigma}^2_{\hat{\theta}} = \sigma^2_{\tilde{\theta}} + \underline{x}^T \Sigma_w \underline{x}$$

$$= E\sigma^2_{\tilde{\theta}} + E\underline{x}^T \Sigma_w \underline{x}$$

$$E\sigma^2_{\hat{\theta}} = \sigma^2_{\tilde{\theta}} + t(\Sigma_w \Sigma_x) + \underline{\mu}_x^T \Sigma_w \underline{\mu}_x$$