A CONSTRUCTIVE SOLUTION FOR TWO-DIMENSIONAL CONTROLLED SELECTION PROBLEMS

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1. INTRODUCTION

Consider a population stratified by two criteria of stratification, r rows and c columns, resulting in a two-way table of rc strata cells. A sample of size n is to be selected, with g_{ij} denoting the expected number of sample units in the ij-th cell. We wish to limit the deviation of the number of sample units in the ij-th cell from g_{ij}, and also limit the deviation for the row and column totals, while strictly maintaining the requirements of probability sampling. This will be done by construction of a set of rxc integer valued arrays, together with associated probabilities of selection, where n_{ijk}, the ij-th entry for the k-th array, is the number of sample units in the ij-th cell if the k-th array is selected, and satisfies the following:

$$E(n_{ijk}|i,j) = g_{ij}, \quad (1.1)$$

$$|n_{ijk} - g_{ij}| < 1, \quad (1.2)$$

$$|n_{i.k} - g_{i.}| < 1, \quad (1.3)$$

$$|n_{.jk} - g_{.j}| < 1. \quad (1.4)$$

Thus n_{ijk} is required to be one of the two integers nearest to g_{ij} if g_{ij} is not an integer, and $n_{ijk} = g_{ij}$ if g_{ij} is an integer, with similar statements holding for the marginal totals.

The above conditions, which can be generalized to dimensions greater than two, are a special case of conditions to be satisfied for the sampling technique known as controlled selection, first described by Goodman and Kish (1950). An example was given in that paper to illustrate the application of controlled selection but no general method to solve such problems was presented. Bryant, Hartley, and Jensen (1960) developed a simple method for approaching the two-way stratification problem that we described above. However, their method does not in general satisfy either (1.1) or (1.2) exactly. Jessen (1970) considered the identical requirements as imposed by (1.1)-(1.4), but presented no general procedure for obtaining a set of arrays that satisfies these conditions. Groves and Hess (1975) presented a formal algorithm for obtaining solutions to the two-dimensional and also the much more complex three-dimensional problem. They made no claim, however, that their algorithm will always yield a solution, and there are indeed simple examples where it fails, even in the two-way case.

In this paper, after establishing notation in Section 2, we describe in Section 3 a constructive method for solving two-dimensional controlled selection problems, which in the Appendix is proved to always yield a solution. (Copies of the Appendix are available from the author.) In Section 4 we show by example that the three-dimensional problem does not always have a solution.

The construction given in this paper also yields a strengthened solution to a related problem, the controlled rounding problem. Employing the terminology of Cox and Ernst (1981), a controlled rounding of a rxc array (b_{ij}) is an integer valued rxc array (a_{ij}) satisfying

$$b_{ij} \le a_{ij} \le b_{ij} + 1, \quad (1.5)$$

$$b_{i.} \le a_{i.} \le b_{i.} + 1, \quad (1.6)$$

$$b_{.j} \le a_{.j} \le b_{.j} + 1, \quad (1.7)$$

$$b_{..} \le a_{..} \le b_{..} + 1. \quad (1.8)$$

In that paper a procedure for constructing a controlled rounding of any two dimensional array was detailed. In the present paper for each k the array (n_{ijk}) to be constructed is by (1.2)-(1.4) not only a controlled rounding of (g_{ij}), but even satisfies a slightly stronger condition since (1.5)-(1.8) would still hold if each right "$\le$" was replaced by "$<$". Thus, for example, if g_{ij} was an integer then $n_{ijk} = g_{ij}$ by (1.2), while (1.5) only guarantees that either $n_{ijk} = g_{ij}$ or $n_{ijk} = g_{ij} + 1$.

2. NOTATION AND TERMINOLOGY

For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x, while $\lceil x \rceil$ is the smallest integer greater than or equal to x.

Let $C = \{1,\ldots,c\}$

A tabular array is an (r+1)x(c+1) array of numbers $B = (b_{ij})$ such that

$$b_{i(c+1)} = b_{i.} = \sum_{j=1}^{c} b_{ij}, \quad i=1,\ldots,r, \quad (2.1)$$

$$b_{(r+1)j} = b_{.j} = \sum_{i=1}^{r} b_{ij}, \quad j=1,\ldots,c, \quad (2.2)$$

$$b_{(r+1)(c+1)} = b_{..} = \sum_{i=1}^{r} \sum_{j=1}^{c} b_{ij}. \quad (2.3)$$

The elements b_{ij}, $i=1,\dots,r$, $j=1,\dots,c$ will be known as the internal elements of the tabular array. They uniquely define the array.

Let G denote the tabular array with internal elements g_{ij}, $i=1,\dots,r$, $j=1,\dots,c$.

3. THE CONSTRUCTIVE ALGORITHM

We will recursively define a finite sequence of integer valued tabular arrays, $N_1=(n_{ij1})$, $N_2=(n_{ij2}),\dots,$ $N_\ell=(n_{ij\ell})$, and associated positive probabilities $p_1,\dots,p_\ell$, satisfying

$$\sum_{k=1}^{\ell} p_k = 1, \qquad (3.1)$$

$$E(n_{ijk}|i,j) = \sum_{k=1}^{\ell} n_{ijk}p_k = g_{ij} \quad \text{for all } i,j, \qquad (3.2)$$

$$\lfloor g_{ij}\rfloor \le n_{ijk} \le \lceil g_{ij}\rceil \text{ for all } i,j,k. \qquad (3.3)$$

The justification for all aspects of the algorithm will be given in the Appendix.

To construct N_k, p_k we begin with the tabular array $A_k = (a_{ijk})$. $A_1 = G$, while for $k > 1$, A_k will be defined at the end of the construction of N_{k-1}, p_{k-1}. Now

$$\lfloor g_{ij}\rfloor \le a_{ijk} \le \lceil g_{ij}\rceil \text{ for all } i,j,k \qquad (3.4)$$

(see Appendix), and consequently for (3.3) to be satisfied it suffices to define N_k with

$$\lfloor a_{ijk}\rfloor \le n_{ijk} \le \lceil a_{ijk}\rceil \text{ for all } i,j,k. \qquad (3.5)$$

We next make the following definitions. (Henceforth, the third subscript will be dropped from a_{ijk} and n_{ijk} if this will cause no confusion.) We initially let

$$n_{ij} = \lfloor a_{ij}\rfloor,\ i = 1,\dots,r,\ j = 1,\dots,c+1,$$

abbreviate

$$R_i = n_{i(c+1)},\ i=1,\dots,r,$$

and define for $S \subset C$,

$$m(S) = \sum_{i=1}^{r} \min\{R_i - \sum_{j\in C\sim S} n_{ij}, \sum_{j\in S} \lceil a_{ij}\rceil\} - \sum_{j\in S} \lfloor a_{.j}\rfloor,$$

$$M(S) = \sum_{i=1}^{r} \max\{R_i - \sum_{j\in C\sim S} \lceil a_{ij}\rceil, \sum_{j\in S} n_{ij}\} - \sum_{j\in S} \lceil a_{.j}\rceil.$$

Note that at this point $n_{(r+1)j}$, $j = 1,\dots,c+1$ has not been defined at all, and that most of the other required conditions are not satisfied by the n_{ij}'s as currently defined. In Steps 1-4 below we remedy this situation. We first proceed in Steps 1 and 2 to redefine, one at a time, some of the R_i's to be $\lceil a_{i.}\rceil$ until we reach the point, upon the completion of Step 2, where $\Sigma_{i=1}^{r} R_i = a_{..}$. Then in Step 3 we redefine, again one at a time, some of the internal elements of N_k to be $\lceil a_{ij}\rceil$, until at the end of Step 3,

$$\sum_{j=1}^{c} n_{ij} = R_i,\quad i=1,\dots,r. \qquad (3.6)$$

Finally, in Step 4 we let

$$n_{(r+1)j} = n_{.j} = \sum_{i=1}^{r} n_{ij},\quad j=1,\dots,c+1. \qquad (3.7)$$

In order that (3.5) hold for $i=r+1$, $j=1,\dots,c$, care must be taken in Steps 1-3 in choosing the elements of N_k to be redefined to guarantee that the final internal elements, determined upon completion of Step 3, satisfy

$$\lfloor a_{.j}\rfloor \le \sum_{i=1}^{r} n_{ij} \le \lceil a_{.j}\rceil,\quad j=1,\dots,c, \qquad (3.8)$$

or equivalently,

$$\sum_{j\in S} \lfloor a_{.j}\rfloor \le \sum_{j\in S}\sum_{i=1}^{r} n_{ij} = \sum_{j\in S} \lceil a_{.j}\rceil,\ S\subset C. \qquad (3.9)$$

To satisfy the first inequality in (3.9) it is necessary that for the final n_{ij}'s,

$$m(S) \ge 0,\quad S\subset C, \qquad (3.10)$$

while to satisfy the second inequality it is necessary that

$$M(S) \le 0,\quad S\subset C. \qquad (3.11)$$

In Step 1 the elements to be increased are chosen with the goal of satisfying (3.10). Once this has been accomplished, the elements to be increased in Step 2 are chosen so that (3.11) remains satisfied, and then in Step 3 so that both (3.10) and (3.11) remain satisfied. Note that in general each time an element is increased in Steps 1-3, the values of the functions m and M are changed for some subsets. (See Appendix for further explanation.)

Step 1: If $m(S) \geq 0$ for all $S \subset C$, proceed to Step 2. Otherwise, choose $S_0 \subset C$ of minimal cardinality satisfying $m(S_0) < 0$. Then choose i_0 with

$$R_{i_0} < \min\left\{\sum_{j \in S_0} \lceil a_{i_0 j}\rceil + \sum_{j \in C \sim S_0} n_{i_0 j},\ \lceil a_{i_0 \cdot}\rceil\right\}, \tag{3.12}$$

and let $R_{i_0} = \lceil a_{i_0 \cdot}\rceil$. Finally, return to the beginning of this step.

Step 2: If $\Sigma_{i=1}^{r} R_i \geq a_{\cdot\cdot}$ proceed to Step 3. Otherwise, let $T = \bigcup\{S: M(S) = 0\}$, choose i_0 for which

$$R_{i_0} < \min\left\{\sum_{j \in C \sim T} \lceil a_{i_0 j}\rceil + \sum_{j \in T} n_{i_0 j},\ \lceil a_{i_0 \cdot}\rceil\right\}, \tag{3.13}$$

and let $R_{i_0} = \lceil a_{i_0 \cdot}\rceil$. Then return to the beginning of this step.

Step 3: Substeps 3A, 3B, and 3C below insure that (3.6), (3.10), and (3.11) respectively are satisfied.

Step 3A: If $\{i: R_i > \Sigma_{j=1}^{c} n_{ij}\}$ is empty, proceed to Step 4. Otherwise, let i_0 denote the minimal element of this set, let

$$J = C \cap \{j: n_{i_0 j} < \lceil a_{i_0 j}\rceil\},$$

and proceed to Step 3B.

Step 3B: If there exists $S \subset C$ for which $m(S) = 0$ and $S \cap J \neq \emptyset$, then choose S_0 with minimal cardinality satisfying these conditions, choose $j_0 \in S_0 \cap J$, and proceed to Step 3D. Otherwise, proceed to Step 3C.

Step 3C: If there exists $S \subset C$ for which $M(S) = 0$ and $J \sim S \neq \emptyset$, then choose S_0 with maximal cardinality satisfying these conditions, and choose $j_0 \in J \sim S_0$. Otherwise, choose any $j_0 \in J$. Proceed to Step 3D.

Step 3D: Let $n_{i_0 j_0} = \lceil a_{i_0 j_0}\rceil$ and return to the beginning of Step 3A.

Step 4: Having obtained the final value of all the elements of N_k except those in the bottom row, we complete the definition of N_k with (3.7).

Step 5: We now proceed to define p_k. First, for each i,j let

$$t_{ijk} = \lfloor a_{ij}\rfloor + 1 - a_{ij} \quad \text{if } n_{ijk} = \lfloor a_{ij}\rfloor,$$

$$= a_{ij} - \lfloor a_{ij}\rfloor \quad \text{if } n_{ijk} = \lfloor a_{ij}\rfloor + 1.$$

Then let d_k denote the minimum value of $\{t_{ijk}: i=1,\ldots,r+1, j=1,\ldots,c+1\}$, and

$$p_k = d_k \qquad \text{if } k = 1.$$

$$= (1 - \sum_{i=1}^{k-1} p_i)d_k \quad \text{if } k > 1. \tag{3.14}$$

Now if $d_k = 1$, then $\Sigma_{i=1}^{k} p_i = 1$ and we are done, that is $N_1,\ldots,N_k$ together with the associated probabilities $p_1,\ldots,p_k$ provide a solution to the controlled selection problem. Otherwise, we define A_{k+1} by letting

$$a_{ij(k+1)} = \frac{a_{ijk} - d_k n_{ijk}}{1 - d_k} \tag{3.15}$$

for all i, j, and then return to the beginning of Step 1. The fact that this process terminates after a finite number of steps, that is, that there exists an integer ℓ for which $d_\ell = 1$, is proven in the Appendix.

4. THREE WAY STRATIFICATION

The following example illustrates that the three-way stratification problem does not always have a solution.

Consider a population subject to a 2x2x2 stratification from which a sample of size two is to be drawn. The expected number of sample units in the ijk-th stratum cell, g_{ijk}, is as follows:

$$g_{111} = g_{221} = g_{122} = g_{212} = 0.5,$$

$$g_{121} = g_{211} = g_{112} = g_{222} = 0.$$

We demonstrate that there is no solution by proving that there exists no 2x2x2 integer valued matrix $N=(n_{ijk})$ such that

$$|n_{ijk} - g_{ijk}| < 1 \quad \text{for all } i,j,k, \tag{4.1}$$

$$|n_{i\cdot\cdot} - g_{i\cdot\cdot}| < 1 \quad \text{for } i=1,2, \tag{4.2}$$

$$|n_{\cdot j\cdot} - g_{\cdot j\cdot}| < 1 \quad \text{for } j=1,2, \tag{4.3}$$

$$|n_{\cdot\cdot k} - g_{\cdot\cdot k}| < 1 \quad \text{for } k=1,2, \tag{4.4}$$

$$n_{\cdot\cdot\cdot} = 2. \tag{4.5}$$

To show this we note that there are six possible N's which will satisfy (4.1) and (4.5). The nonzero elements of N for each of these possibilities are as follows: $n_{111} = n_{221} = 1$, $n_{111} = n_{122} = 1$, $n_{111} = n_{212} = 1$, $n_{221} = n_{122} = 1$, $n_{221} = n_{212} = 1$ and $n_{122} = n_{212} = 1$. The first possibility fails since $n_{\cdot\cdot 1} = 2$ and $g_{\cdot\cdot 1} = 1$, and hence (4.4) is not satisfied. Similarly, the others fail to satisfy (4.2), (4.3), (4.3), (4.2), and (4.4) respectively.

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