AN ADDITIVE RANDOMIZED RESPONSE MODEL

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I. INTRODUCTION

In a number of empirical situations, observations can be coded in order to insure privacy or non-disclosure of the actual values. This coding can be performed by the addition or multiplication of a random variable of known distribution and we will term models of this type augmentation models. In surveying human populations, Warner (1971) suggested that this approach might be used in randomized response technique (RRT) models, and we present in this paper an additive one. Initially, we will present the additive model for the three category case, and then generalize it to the k category case. Some comparisons with potentially competing RRT models will be briefly given. Finally, some results from a small field trial will be presented. Before we present the additive model, we will briefly review Warner's two question model (1965).

Warner (1965) presented a model for estimating the population proportion (π) of people who possess sensitive characteristic 1 in the given population. The model requires the respondent to use a randomization device in order to determine which one of the two questions is answered. The two questions are of the form:

Q1. I am a member of group 1 (G_1),

Q2. I am a member of group Not-1 ($G_{\bar{1}}$),

and the randomization device selects Q1 with probability p and Q2 with probability (1-p). Hence the probability of a "yes" answer under the assumption of truthful reporting is

$$\lambda = \Pr(\text{yes}) = p\pi+(1-p)(1-\pi) \qquad (1.1)$$

Assuming that the likelihood function of the number of "yeses" (Z) in a sample of size n is binomially distributed with probability λ, Warner obtained the estimator of π as

$$\hat{\pi} = \{Z/n-(1-p)\}/(2p-1) \qquad (1.2)$$

He also presented the variance of $\hat{\pi}$

$$V_1(\hat{\pi}) = \frac{\pi(1-\pi)}{n} + \frac{p(1-p)}{n(2p-1)^2} \quad (p\neq\tfrac{1}{2}) \qquad (1.3)$$

II. THE ADDITIVE MODEL

The additive (A) RRT model presupposes that in response to a direct question a respondent can truthfully classify himself as a member of one, and only one, of k groups. However, the respondent is reluctant to do so in light of the sensitive nature of the question. As such, the respondent is asked to code (augment) his answer in order to achieve a measure of confidentiality.

2.1. The Case for k=3

Let C_j be the true group classification (number) of the j^{th} respondent (C_j=1,2 or 3 and j=1,2,...,n), and a_j be his randomly selected augmentation value (a_j=1,2 or 3). Then the j^{th} respondent's coded response (y_j) whose true group is C_j, is

$$y_j = C_j+a_j, \quad C_j=1,2,3; \quad j=1,2,\ldots,n \qquad (2.1)$$

To provide further confidentiality to the respondent, y_j is transformed by the respondent to the reported value, R_j.

$$R_j = \begin{cases} y_j & \text{if } y_j\le 3 \\ y_j-3 & \text{if } y_j>3 \end{cases} \qquad j=1,2,\ldots,n.$$

Table 1 presents the possible reported values and their sources (C+a). For convenience, we drop the subscipt j.

Table 1

The Response Transformations for the Additive Model

Reported Number (R)	Source (C+a)		
1	1+3	2+2	3+1
2	1+1	2+3	3+2
3	1+2	2+1	3+3

We define $\pi_C = \Pr(x\epsilon C)$, C=1,2,3, $P_a = \Pr$(x selecting a), a=1,2,3, where x represents any respondent.

Following Table 1, the probability (λ_R) that a person reports value R(R=1,2,3) is:

$$\lambda_1 = p_3\pi_1 + p_2\pi_2 + p_1\pi_3$$
$$\lambda_2 = p_1\pi_1 + p_3\pi_2 + p_2\pi_3$$
$$\lambda_3 = p_2\pi_1 + p_1\pi_2 + p_3\pi_3$$

Noting that $\lambda_3 = 1-\lambda_1-\lambda_2$ and $\pi_3 = 1-\pi_1-\pi_2$, the equations of interest reduce to,

$$\lambda_1 = p_1 + (p_3-p_1)\pi_1 + (p_2-p_1)\pi_2$$
$$\lambda_2 = p_2 + (p_1-p_2)\pi_1 + (p_3-p_2)\pi_2$$

In matrix notation,

$$\underline{\lambda}^* = P\underline{\pi},$$

where $\underline{\lambda}^{*\prime} = (\lambda_1-p_1,\lambda_2-p_2)$, $\underline{\pi}' = (\pi_1,\pi_2)$ and

$P = \begin{pmatrix} p_3-p_1 & p_2-p_1 \\ p_1-p_2 & p_3-p_2 \end{pmatrix}$. Given that $|P|\neq 0$ (note; $|P|=0$ iff $p_1=p_2=p_3=1/3$),

$$\hat{\underline{\pi}} = P^{-1}\hat{\underline{\lambda}}^* \quad \text{and}$$

$V(\hat{\underline{\pi}}) = P^{-1}V(\underline{\lambda}^*)(P^{-1})'$, where

$$V(\hat{\underline{\lambda}}^*) = \frac{1}{n}\begin{pmatrix} \lambda_1(1-\lambda_1) & -\lambda_1\lambda_2 \\ -\lambda_1\lambda_2 & \lambda_2(1-\lambda_2) \end{pmatrix},$$

$\hat{\underline{\lambda}}^{*\prime} = (\hat{\lambda}_1 - p_1, \hat{\lambda}_2 - p_2)$, $\hat{\lambda}_i = \frac{Z_i}{n}$, and Z_i is a r.v. designating the number of persons who reported themselves as members of the i^{th} group in a sample of $n(Z_1+Z_2+Z_3 = n)$ respondents.

Specifically, the point estimators of π_i, $i=1,2,3$, are:

$$\hat{\pi}_1 = \frac{1}{|P|}[(p_3-p_2)(\hat{\lambda}_1-p_1)+(p_1-p_2)(\hat{\lambda}_2-p_2)], \tag{2.2}$$

$$\hat{\pi}_2 = \frac{1}{|P|}[(p_2-p_1)(\hat{\lambda}_1-p_1)+(p_3-p_1)(\hat{\lambda}_2-p_2)], \tag{2.3}$$

$$\hat{\pi}_3 = 1-\hat{\pi}_1-\hat{\pi}_2. \tag{2.4}$$

These estimators are unbiased and the variances can be shown to be,

$$V(\hat{\pi}_1) = [(p_3-p_2)^2\lambda_1(1-\lambda_1)-2(p_3-p_2)(p_1-p_2)\lambda_1\lambda_2 + (p_1-p_2)^2\lambda_2(1-\lambda_2)]/n|P|^2, \tag{2.5}$$

$$V(\hat{\pi}_2) = [(p_2-p_1)^2\lambda_1(1-\lambda_1)-2(p_2-p_1)(p_3-p_1)\lambda_1\lambda_2 + (p_3-p_1)^2\lambda_2(1-\lambda_2)]/n|P|^2, \tag{2.6}$$

$$V(\hat{\pi}_3) = [(p_3-p_1)^2\lambda_1(1-\lambda_1)-2(p_3-p_1)(p_3-p_2)\lambda_1\lambda_2 + (p_3-p_2)^2\lambda_2(1-\lambda_2)]/n|P|^2, \tag{2.7}$$

with

$$\mathrm{Cov}(\hat{\pi}_1,\hat{\pi}_2) = [(p_3-p_2)(p_2-p_1)\lambda_1(1-\lambda_1)+(p_2-p_1)^2\lambda_1\lambda_2 -(p_3-p_2)(p_3-p_1)\lambda_1\lambda_2+(p_1-p_2)(p_3-p_1)\lambda_2(1-\lambda_2)]/n|P|^2 \tag{2.8}$$

2.2. The General Case

This additive model is easily extendable to the case of any finite (k) number of groups. The coded responses are again given by equation 2.1, where C=1,2,...,k and a=1,2,...,k. Thus, the reported value for respondent j is,

$$R_j = \begin{cases} y_j & \text{if} \quad y_j \le k \\ y_j - k & \text{if} \quad y_j > k \qquad j=1,2,\ldots,n. \end{cases}$$

Noting the constraint $\sum_{C=1}^{k} \pi_C = 1$, the λ's of interest are:

$$\begin{aligned}
\lambda_1 &= p_1 + (p_k-p_1)\pi_1 + \ldots + (p_2-p_1)\pi_{k-1}, \\
\lambda_2 &= p_2 + (p_1-p_2)\pi_1 + \ldots + (p_3-p_2)\pi_{k-1}, \\
&\vdots \\
\lambda_{k-1} &= p_{k-1} + (p_{k-2}-p_{k-1})\pi_1 + \ldots + (p_k-p_{k-1})\pi_{k-1}.
\end{aligned}$$

Defining $\underline{\lambda}^{*\prime} = (\lambda_1-p_1, \lambda_2-p_2, \ldots, \lambda_{k-1}-p_{k-1})$, $\underline{\pi}' = (\pi_1, \pi_2, \ldots, \pi_{k-1})$ and

$$P = \begin{bmatrix} p_k-p_1 & p_{k-1}-p_1 & \cdots & p_2-p_1 \\ p_1-p_2 & p_k-p_2 & \cdots & p_3-p_2 \\ \vdots & & & \\ p_{k-2}-p_{k-1} & p_{k-3}-p_{k-1} & \cdots & p_k-p_{k-1} \end{bmatrix}$$

then $\underline{\lambda}^* = P\underline{\pi}$.

Assuming $|P| \neq 0$, $\hat{\pi} = P^{-1}\hat{\underline{\lambda}}^*$, where $\hat{\underline{\lambda}}^* = (\hat{\lambda}_1-p_1, \hat{\lambda}_2-p_2, \ldots, \hat{\lambda}_{k-1}-p_{k-1})'$. Then $E(\hat{\underline{\pi}}) = P^{-1}\underline{\lambda}^*$, and $V(\hat{\underline{\pi}}) = P^{-1}V(\hat{\underline{\lambda}}^*)(P^{-1})'$, where $V(\hat{\underline{\lambda}}^*) = \{\sigma_{ij}\}$, $\sigma_{ii} = \frac{\lambda_i(1-\lambda_i)}{n}$ and $\sigma_{ij} = -\frac{\lambda_i\lambda_j}{n}$, $i,j=1,2,\ldots,k-1$.

It is interesting to note that the Warner (1965) contamination model is simply a special case of the additive model (k=2), and hence the same holds for the original Warner model.

III. COMPARISONS OF FOUR TRICHOTOMOUS RRT MODELS

Three known published models also are available for this trichotomous sampling estimation problem, and we briefly review these three models.

3.1. Warner's Two-Fold Model

Warner's two-fold (TF) model is simply the application of the original Warner model (1965) twice to the same group of respondents. The first application is for estimating π_1 and the second for estimating π_2. The estimator of π_i $(i=1,2)$ can be obtained by replacing p by p_i in (1.2). Its variance also can be found by substituting π_i and p_i for π and p, respectively, in (1.3), i=1,2.

It can be easily shown that $\mathrm{Cov}(\hat{\pi}_1,\hat{\pi}_2) = -\frac{\pi_1\pi_2}{n}$.

3.2. Extended Contamination Model

The extended contamination (EC) model is an extended version of Warner's original contamination model (1965).

Table 2 gives the set of all possible "design matrices," where 1 and 0 represent "yes" and "no," respectively. Matrices (2) to (6) are formed by permutting columns of matrix (1).

Table 2

The Response Transformations for the Extended Contamination Model

	1	2	3	4	5	6
Reported Group / Real Group	123	123	123	123	123	123
1	100	100	001	010	001	010
2	010	001	010	100	100	001
3	001	010	100	001	010	100

The design matrices for this model designates when the respondent reports his true group and when he reports an untrue group (contaminated response). For example, if a respondent randomly selects matrix 2, and if he belongs to group 2, then he has to report "3".

From Table 2, p_i = Pr(person in group i is asked to report R) = 1/3, R=1,2,3, and hence the probability of reporting group R(R=1,2) is

$$\lambda_1 = \frac{1}{3}\pi_1 + \frac{2}{3}(1-\pi_1), \text{ and } \lambda_2 = \frac{1}{3}\pi_2 + \frac{2}{3}(1-\pi_2) \quad (3.1)$$

The form is familiar (a special case of the TF model) and it is easily seen that the estimators and their variances of this model are found by substitution in equations (1.2) and (1.3). The $Cov(\hat{\pi}_1,\hat{\pi}_2)$ can be found to be

$$Cov(\hat{\pi}_1,\hat{\pi}_2) = -\frac{1+\pi_1\pi_2}{n} \quad (3.2)$$

3.3. Multiproportions Model

The third model of interest is Abul-Ela, et al.'s multiproportions (MP) model (1967). This model utilizes two subsamples and the question structure is:

Q1. I am a member of group 1.
Q2. I am a member of group 2.
Q3. I am a member of group 3.

In subsample i, i=1,2, Q1 is selected with probability p_{i1}, Q2 with p_{i2} and Q3 with p_{i3} $\sum_{j=1}^{3} p_{ij} = 1$. If the selected statement is true, the respondent says "yes." The general expression for λ_i is

$$\lambda_i = (p_{i1}-p_{i3})\pi_1 + (p_{i2}-p_{i3})\pi_2 + p_{i3}, \quad i=1,2.$$

Estimators of π's and their variances are

$$\hat{\pi}_1 = \frac{(\hat{\lambda}_1-p_{13})(p_{22}-p_{23})-(\hat{\lambda}_2-p_{23})(p_{12}-p_{13})}{A}, \quad (3.3)$$

$$\hat{\pi}_2 = -\frac{(\hat{\lambda}_1-p_{13})(p_{21}-p_{23})-(\hat{\lambda}_2-p_{23})(p_{11}-p_{13})}{A}, \quad (3.4)$$

and $\hat{\pi}_3 = 1-\hat{\pi}_1-\hat{\pi}_2$, where $A = (p_{11}-p_{13})(p_{22}-p_{23}) - (p_{21}-p_{23})(p_{12}-p_{13})$, and

$$V(\hat{\pi}_1) = \frac{1}{A^2}\left[(p_{22}-p_{23})^2\frac{\lambda_1(1-\lambda_1)}{n_1}+(p_{12}-p_{13})^2\frac{\lambda_2(1-\lambda_2)}{n_2}\right] \quad (3.5)$$

$$V(\hat{\pi}_2) = \frac{1}{A^2}\left[(p_{21}-p_{23})^2\frac{\lambda_1(1-\lambda_1)}{n_1}+(p_{11}-p_{13})^2\frac{\lambda_2(1-\lambda_2)}{n_2}\right] \quad (3.6)$$

$$V(\hat{\pi}_3) = \frac{1}{A^2}\left[(p_{22}-p_{21})^2\frac{\lambda_1(1-\lambda_1)}{n_1}+(p_{12}-p_{11})^2\frac{\lambda_2(1-\lambda_2)}{n_2}\right] \quad (3.7)$$

and

$$Cov(\hat{\pi}_1,\hat{\pi}_2) = \left[\frac{1}{A^2}(p_{22}-p_{23})(p_{23}-p_{21})\frac{\lambda_1(1-\lambda_1)}{n_1} + (p_{12}-p_{13})(p_{13}-p_{11})\frac{\lambda_2(1-\lambda_2)}{n_2}\right] \quad (3.8)$$

where n_i is the size of subsample i, i=1,2.

3.4. The Comparisons of the Models

Four trichotomous RRT models (A,TF,EC, and MP) briefly are compared for the same values of p utilizing a sample of n=100. Since the EC model presented here requires all p's equal to 1/3, these four models are first compared for p_i=1/3, i=1,2,3, and then the three other models are compared for $p_i \neq 1/3$, i=1,2,3. A number of criteria are available for comparing models (Raghavarao, 1971, Ch. 17), and because all the estimators involved are unbiased, we will use the criterion $V(\pi_1) + V(\pi_2)$ as the measure of evaluation. With the exception of the MP model, all models are one sample models, and for the MP model it will be assumed, for convenience, that the two subsample sizes are equal ($n_1=n_2$).

When p_i=1/3, i=1,2,3, models MP and A have variances which are undefined and models TF and EC have the same estimators and variances. However, the EC model requires a single trial while the TF model requires two trials (essentially, the TF model uses twice as large sample size as the EC model), and hence in terms of cost and time of performing sampling, the EC model is preferred.

Table 3 presents some numerical examples of the variances of the three trichotomous models (A,TF and MP) for $p_i \neq 1/3$, i=1,2,3, π_3=.025, .05, .10 and varying π_1 and π_2. It should be mentioned that in the table, 999 signifies an undefined quantity and p_{11} and p_{12} are p_1 and p_2 respectively, in the one sample models (A and TF). In most cases, the A model has the least variance. When p_1=.7 and p_2=.1, the TF model also has relatively small variances. In short, the A model generally is best, and the performance of the MP model generally is erratic.

Leysieffer and Warner (1976) also presented a measure of respondent protection. They argue that the choice of best design should be made only among designs that have the same level of respondent jeopardy, as measured by K_i. Kim (1978) presents some results using this measure of protection and in general the A model again is best.

IV. FIELD TRIAL OF THE ADDITIVE MODEL

The additive model and Abul-Ela, et al.'s multiproportions model (Section III) were field-tested in a small trial. A single sensitive characteristic (i.e., cheating on a course exam at a university) was chosen.

The questions were typed on a single sheet of paper and phrased as follows:

Q1. I have never thought of cheating.

Q2. I had prepared for cheating before the test but did not actually cheat.

Q3. I cheated.

The field trial was designed such that everyone in the sample used both RRT models; half of the respondents used the additive model first and the other half used the multiproportions model first. After each trial, the respondent was asked to mention which model was easier for him to use and which model he believed gave him better protection.

Two classes of Temple statistics students (i.e., Stat. 11 and Stat. 2) and some Temple students present on the campus (around 1/5 of the total sample) were selected in the fall of 1977 for the field trial (n=50). For the multiproportions model, they were divided into two equal groups ($n_1=n_2=25$). Before interviewing a respondent, the research problem of interest was explained as well as how to select and answer a question. An example using a question other than the test question always was presented. It should be noted that in the classroom situation, the interviewer traveled from seat to seat in order to make the procedure as similar to the real interview as possible. No one refused to cooperate in this trial.

Table 4 presents the sample results for the additive and multiproportions model, respectively.

Table 3

Variances for Comparisons of Three Trichotomous Models Based on $p \neq 1/3$

π_1	π_2	π_3	P_{11}	P_{12}	P_{21}	P_{21}	A	TF	MP
.925	.050	.025	.70	.10	.10	.70	.0103	.0157	.0068
					.30	.50	.0103	.0157	.0278
					.50	.30	.0103	.0157	.8086
.900	.075	.025	.70	.10	.10	.70	.0106	.0161	.0073
					.30	.50	.0106	.0161	.0280
					.50	.30	.0106	.0161	.8085
.80	.15	.05	.70	.10	.10	.70	.0111	.0174	.0086
					.30	.50	.0111	.0174	.0286
					.50	.30	.0111	.0174	.08055
.70	.20	.10	.70	.10	.10	.70	.0114	.0182	.0094
					.30	.50	.0114	.0182	.0289
					.50	.30	.0114	.0182	.7972
.60	.30	.10	.70	.10	.10	.70	.0114	.0190	.0105
					.30	.50	.0114	.0190	.0297
					.50	.30	.0114	.0190	.7869

Table 4

The Sample Responses for the Additive Model and the Multiproportions Model

A. Additive

Group Number Reported	1	2	3
Number Reporting	14	20	16

B. Multiproportions

	Subsample Number	
	1	2
Number Reporting "True"	6	5
Number Reporting "False"	19	20

Using equations (2.2) and (2.3), the additive model estimates are:

$$\hat{\pi}_1 = \frac{1}{(\frac{2}{10}-\frac{5}{10})(\frac{2}{10}-\frac{3}{10})+(\frac{5}{10}-\frac{3}{10})^2}\left[(\frac{2}{10}-\frac{3}{10})(\frac{14}{50}-\frac{5}{10})\right.$$

$$\left.+(\frac{5}{10}-\frac{3}{10})(\frac{20}{50}-\frac{3}{10})\right] = \frac{100}{7}(\frac{11+10}{500}) = \frac{3}{5} = .60$$

$$\hat{\pi}_2 = \frac{100}{7}\left[(\frac{3}{10}-\frac{5}{10})(\frac{14}{50}-\frac{5}{10}) + (\frac{2}{10}-\frac{5}{10})(\frac{20}{50}-\frac{3}{10})\right]$$

$$= \frac{100}{7}(\frac{22-15}{500}) = \frac{1}{5} = .20$$

Hence $\hat{\pi}_3=.20$. Using equations (2.5), (2.6) and (2.7), the corresponding estimates of the variances are:

$v(\hat{\pi}_1)=.06570$, $v(\hat{\pi}_2)=.06622$ and $v(\hat{\pi}_3)=.05643$

Using equations (3.3) and (3.4), the multiproportions estimates of π_i's, i=1,2,3, are:

$$\hat{\pi}_1 = \frac{(\frac{6}{25}-\frac{2}{10})(\frac{2}{10}-\frac{1}{10}) - (\frac{5}{25}-\frac{1}{10})(\frac{3}{10}-\frac{2}{10})}{-\frac{3}{100}} = .20$$

$$\hat{\pi}_2 = -\frac{(\frac{6}{25}-\frac{2}{10})(\frac{7}{10}-\frac{1}{10}) - (\frac{5}{25}-\frac{1}{10})(\frac{5}{10}-\frac{2}{10})}{-\frac{3}{100}}$$

$$= -\frac{-\frac{3}{500}}{-\frac{3}{100}} = -.20.$$

and hence $\hat{\pi}_3=1.00$. Using equations (3.5), (3.6) and (3.7), the corresponding estimates of the variances are:

$v(\hat{\pi}_1)=.15218$, $v(\hat{\pi}_2)=4.056$ and $v(\hat{\pi}_3) = 2.14$.

The additive model estimates of π_i's, i=1,2,3, seem plausible, whereas the multiproportions model estimates of π_i's, i=1,2,3, are not. They clearly require some modification.

Table 5 provides the respondent results (in percentage) on the questions of the ease of use and the perceived protection of the two RRT models.

Table 5

The Sample Responses on Ease and Perceived Protection for the Additive and the Multiproportions Models

	Prefer Additive Model	Prefer Multiprop. Model	No Preference
Easier	4	86	10
Better Protection	30	56	14

With respect to "ease of use", the results show that the respondents generally considered the multiproportions model easier to use (86%). Their principal reason for choosing the multiproportions model appeared to be that the additive model required additional coding of answers by the respondent (see Section II), whereas the multiproportions model did not. Perhaps the use of "cue cards" or a card with all possible combinations (Table 1) would result in easier use of the additive model.

V. CONCLUSION

In this paper an additive RRT model was presented for use in surveying either human or non-human populations. To evaluate it's relative strengths, it was compared with Abul-Ela, et al.'s multiproportions model, a two-fold version of the original Warner model and Warner's extended contamination model. When the probability of selecting each of three questions (p_i, $i=1,2,3$) is the same, the A and MP models have undefined variances, and the EC model and the TF model have the same variance as a function of p. Considering the number of trials needed to get the same information, the EC model is considered better than the TF model. When $p_i \neq 1/3$, $i=1,2,3$, the A model is generally best.

The field trial results show that the additive model is usable, but apparently not as easy to use as the multiproportions model. In this regard, we believe that a cue card with the alternative answers (Table 1) would be very helpful and make the additive model easier to use. Lastly, it should be noted that the MP model gave a negative estimate of π_2.

REFERENCES

[1] Abul, Ela, A.A., Greenberg, B.G., and Horvitz, D.G., "A Multiproportions Randomized Response Model," Journal of the American Statistical Association, 62 (1967), 990-1008.

[2] Kim, J., "Randomized Response Techniques for Surveying Human Populations," Ph.D. Dissertation, Department of Statistics, Temple University, Philadelphia, Pa. (1978), 1-144.

[3] Leysieffer, F.L., and Warner, S.L., "Respondent Jeopardy and Optimal Designs in Randomized Response Models," Journal of the American Statistical Association, 71 (1976), 649-656.

[4] Raghavarao, D., Constructions and Combinatorial Problems in Designs of Experiments, Wiley & Sons, Inc., 1971.

[5] Warner, S.L., "Randomized Response: A Survey Technique for Eliminating Evasive Answer Bias," Journal of the American Statistical Association, 60 (1965), 63-69.

[6] Warner, S.L., "Linear Randomized Response Models," Journal of the American Statistical Association, 66 (1971), 884-888.